Wednesday, November 15, 2023 12:09 PM

GIVEN $v \in \mathbb{R}^n$

HOW TO COMPUTE $L_x(v)$???

GIVEN $v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$

$$\overset{!}{=} \underline{\sum_{i=1}^{n} v_i \cdot e_i} \qquad (*)$$

BUT, BY $(*)$

$$\underline{L_x(v)} = L_x\left(\sum_{i=1}^{n} v_i \cdot e_i\right) \overset{\text{LIN}}{=} \sum_{i=1}^{n} v_i \cdot L_x(e_i) \overset{\text{THM}}{=}$$

$$= \sum_{i=1}^{n} v_i \frac{\partial f}{\partial e_i}(x) = \underline{\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x) \cdot v_i} \qquad (+)$$

PARTIAL DER !!!

MORE COMPACT FORM:

$$L_x(v) \overset{!}{=} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x) \cdot v_i \qquad (+)$$

DEFINE $\operatorname{grad} f(x) \overset{\text{DEF}}{=} \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \dots, \frac{\partial f}{\partial x_n}(x)\right) \in \mathbb{R}^n$

GRADIENT OF f AT x

$(+)$ BECOMES

$$L_x(v) = \langle \operatorname{grad} f(x), v \rangle !$$

LET $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 y + x y^3$ POLYNOMIAL

$\Downarrow$?

DIFFERENTIABLE

WHY? (BY THE TOT. DIFF. THM)

LET $x = \underline{(-1, 1)} \in \mathbb{R}^2$, $v = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \Rightarrow \|v\| = 1$

COMPUTE $\frac{\partial f}{\partial v}(x)$!!!

$$\frac{\partial f}{\partial x}(x,y) = 2xy + y^3 \Big/_{(x,y)=(-1,1)} = -2 + 1 = -1$$

$$\frac{\partial f}{\partial y}(x,y) = x^2 + 3xy^2 \Big/_{(x,y)=(-1,1)} = 1 - 3 = -2$$

$$\Rightarrow \operatorname{grad} f(x) = (-1, -2) \overset{\text{THM}}{\Longrightarrow}$$

$$\Rightarrow \frac{\partial f}{\partial v}(\underline{x}) = \left\langle (-1,-2), \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \right\rangle = \frac{1}{2} - \sqrt{3}$$

THM f DIFFERENTIABLE AT $\underline{x} \in A$ $\Rightarrow$ f CONTINUOUS AT $\underline{x} \in A$.

PROOF THESIS IS

$$f \text{ CONTINUOUS AT } \underline{x} \in A \overset{\text{DEF}}{\Longleftrightarrow} |f(\underline{x}+h) - f(\underline{x})| \xrightarrow[h \to \underline{0} \in \mathbb{R}^n]{} 0$$

$\underset{=}{\text{HP}}$ f DIFFERENTIABLE AT $\underline{x} \in A$ $\overset{\text{DEF}}{\Longleftrightarrow}$

$\exists L_{\underline{x}} : \mathbb{R}^n \to \mathbb{R}$ LINEAR SUCH THAT

$$\lim_{h \to \underline{0} \in \mathbb{R}^n} \frac{f(\underline{x}+h) - f(\underline{x}) - L_{\underline{x}}(h)}{\|h\|} = 0$$

REMEMBER THAT, BY SETTING

$$\underset{\text{ERROR}}{E_{\underline{x}}}(h) \overset{\text{DEF}}{=} f(\underline{x}+h) - f(\underline{x}) - L_{\underline{x}}(h)$$

$$\Updownarrow$$

$$f(\underline{x}+h) - f(\underline{x}) = E_{\underline{x}}(h) + L_{\underline{x}}(h) \qquad (\ddagger)$$

WHERE, BY DIFF COND: $\frac{E_{\underline{x}}(h)}{\|h\|} \xrightarrow[h \to \underline{0} \in \mathbb{R}^n]{} 0$

$$\Downarrow \not\Uparrow$$

$$\left(E_{\underline{x}}(h) \xrightarrow[h \to \underline{0} \in \mathbb{R}^n]{} 0 \right) \Leftarrow (***)$$

BY $(\ddagger)$

$$f(\underline{x}+h) - f(\underline{x}) = E_{\underline{x}}(h) + L_{\underline{x}}(h)$$

$$\Downarrow \qquad \qquad \nearrow^{0}_{h \to 0} \qquad \nearrow^{0}_{h \to 0}$$

$$0 \leq |f(\underline{x}+h) - f(\underline{x})| \leq |E_{\underline{x}}(h)| + |L_{\underline{x}}(h)|$$

BUT NOW, $\underset{(***)}{\text{DIFF}} \Rightarrow$ $E_{\underline{x}}(h) \xrightarrow[h \to \underline{0}]{} 0 \Longleftrightarrow \underline{|E_{\underline{x}}(h)| \xrightarrow[h \to \underline{0}]{} 0}$

WHAT ABOUT

$$|L_{\underline{x}}(\underline{h})| \xrightarrow[\underline{h}\to\underline{0}]{} ???$$

BUT $\quad L_{\underline{x}}(\underline{h}) \doteq \langle \text{grad} f(\underline{x}), \underline{h} \rangle$

$$|L_{\underline{x}}(\underline{h})| = |\langle \text{grad} f(\underline{x}), \underline{h} \rangle| \overset{\text{CAUCHY/SCHWARTZ}}{\leq} \underbrace{\underset{\text{COST}}{\|\text{grad} f(\underline{x})\|} \cdot \underset{\substack{\downarrow \underline{h}\to\underline{0} \\ 0}}{\|\underline{h}\|}}_{\substack{\downarrow \underline{h}\to\underline{0} \\ 0}}$$

IN PLAIN WORDS

THEN

$$0 \leq \underset{\substack{\downarrow \underline{h}\to\underline{0} \\ 0}}{|f(\underline{x}+\underline{h}) - f(\underline{x})|} \leq \underset{\substack{\downarrow \underline{h}\to\underline{0} \\ 0}}{|E_{\underline{x}}(\underline{h})|} + \underset{\substack{\downarrow \underline{h}\to\underline{0} \\ 0}}{|L_{\underline{x}}(\underline{h})|}$$

THAT IS : f CONTINUOUS AT $\underline{x} \in A$ $\quad$ THESIS

QED $\boxed{\text{B}}$

SUFFICIENT COND FOR A FUNCT $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$ TO BE DIFFERENTIABLE AT $\underline{x} \in A$

THM (TOTAL DIFFERENTIAL THM)

LET $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$

ASSUME THAT

HP1 $\quad \exists\, I(\underline{x}, \delta)$, $\delta \in \mathbb{R}^+$ SUCH THAT

FOR ANY $x \in I(\underline{x}, \delta)$ EXIST ALL THE PARTIAL DERIVATIVES AT $x \in I(\underline{x}, \delta)$

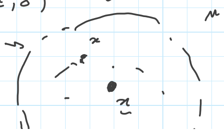
THAT IS

$$\exists\ \frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x) \quad \forall x \in I(\underline{x}, \delta)$$

FOR ANY $x \in I(\underline{x}, \delta)$

$n = 2$

$I(\underline{x},\delta) \to$



$\exists$ BOTH

∂f

$\frac{\partial f}{\partial x}(x)$

$\frac{\partial f}{\partial y}(x)$

HP2 REGARDING

$\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}$ AS FUNCTS ON $I(\underline{x}, \delta)$

(BY HYP 1)

THEY ARE CONTINUOUS AT $\underline{x} \in A$

THEN

TH f DIFFERENTIABLE AT $\underline{x} \in A$.

FIRST APPL.

$f: \mathbb{R}^n \to \mathbb{R}$ POLYNOMIAL

THE PARTIAL DERIVATIVES ARE POLYNOMIAL (THEY EXIST EVERYWHERE

$\Downarrow$

HP1 OK

CONT

$\Downarrow$

HP2

THEN

TOT DIFF THEOREM SAYS THAT

f POLYNOMIAL $\Rightarrow$ f DIFF AT ANY POINT $\underline{x} \in \mathbb{R}^n$

BYE BYE QUESTIONS?