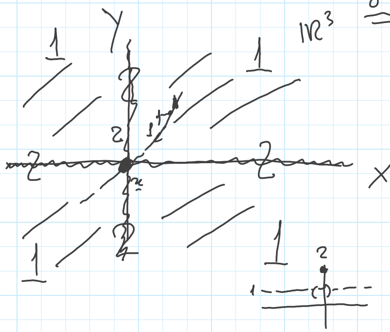
Wednesday, November 15, 2023 12:10 PM

COUNTEREXAMPLE

LET $F: \mathbb{R}^2 \to \mathbb{R}$ SUCH THAT

$$F(x,y) = \begin{cases} 1 & xy \neq 0 \\ 2 & xy = 0 \end{cases}$$



$\mathbb{R}^3 \xrightarrow{F} \mathbb{R}$

$\underline{x} = (0,0)$

WE HAVE

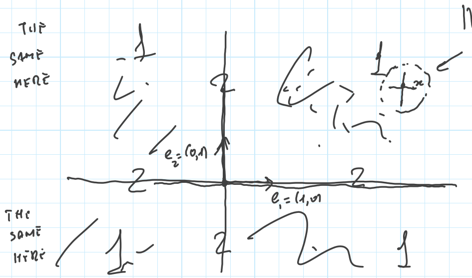
$F(\underline{x}) = F(0,0) = 2$!!!

IS F CONTINUOUS AT $\underline{x} = (0,0)$??? NO !!!

$\Rightarrow$ F IS NOT DIFFERENTIABLE AT $\underline{x} = (0,0)$

$\Rightarrow$ THE HPS OF THE TOTAL DIFFERENTIAL THMS ARE NOT SATISFIED (???)

CONSIDER, AS A GENERAL FACT, THE PARTIAL DERIVATIVES OF THE FUNCT. F !!



$\exists \frac{\partial f}{\partial x}(\underline{x}) = 0$

$\exists \frac{\partial f}{\partial y}(\underline{x}) = 0$

THE SAME HERE

WHAT ABOUT PART. DERIVATIVES AT $\underline{x} = (0,0)$?

? $\exists$ FINITE $\frac{\partial f}{\partial x}(\underline{x}) = \lim_{t \to 0} \frac{f(\underline{x} + t e_1) - f(\underline{x})}{t} = 0$ YES

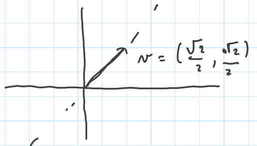
? $\exists$ FINITE $\frac{\partial f}{\partial y}(\underline{x}) = \lim \frac{f(\underline{x} + t e_2) - f(\underline{x})}{} = 0$ YES

$\partial y \qquad t \to 0 \qquad t$

NOTICE THAT:

$$\exists \ \text{grad} f(\underline{x}) = (0,0) \in \mathbb{R}^2 \qquad \underline{x} = (0,0)$$

FIX A DIRECTION, SAY $v = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ S.T $\|v\| = 1$



WHAT ABOUT $\dfrac{\partial F}{\partial v}(\underline{x})$? $\qquad \underline{x} = (0,0)$.

OK

$$\frac{\partial F}{\partial v}(\underline{x}) \overset{\text{THM ???}}{=} \langle \text{grad} F(\underline{x}), v \rangle = \langle (0,0), v \rangle = \underline{0} \ !!! \qquad \left(\begin{array}{l}\text{INDEED} \\ \text{FOR ANY} \\ \text{DIRECTION } v\end{array}\right)$$

IS IT TRUE/CORRECT ??? COMPLETELY FALSE !!!

AS A CONCRETE ANALYSIS, GIVEN $v = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

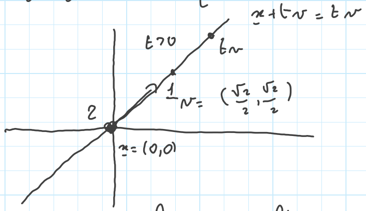
DOES $\dfrac{\partial F}{\partial v}(\underline{x})$ EXIST ??? $\qquad \underline{x} = (0,0)$

IF IT EXISTS, IT MEANS THAT

$$\lim_{t \to 0} \frac{f(\underline{x} + tv) - f(\underline{x})}{t} \quad \text{EXISTS + FINITE}$$

$\underline{x} = (0,0)$

$F(\underline{x}) = 2$



$$\exists \lim_{t \to 0} \frac{f(\underline{x} + tv) - f(\underline{x})}{t} \iff$$

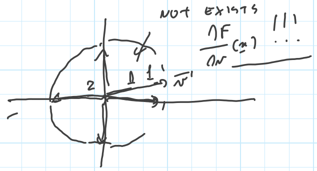
$$\iff \lim_{t \to 0^+} \frac{f(\underline{x} + tv) - f(\underline{x})}{t} \overset{?}{=} \lim_{t \to 0^-} \frac{f(\underline{x} + tv) - f(\underline{x})}{t}$$

$f(\underline{x} + tv) - f(\underline{x}) \qquad 1 - 2 \qquad !!!$

$$\text{BUT} \quad \lim_{t \to 0^+} \frac{\quad}{t} = \lim_{t \to 0^+} \frac{\quad}{t} = -\infty \; !!!$$

$$\text{BUT} \quad \lim_{t \to 0^-} \frac{f(x + t v) - f(x)}{t} = \lim_{t \to 0^-} \frac{1 - 2}{t} = +\infty \; !!!$$

$$\underline{\text{THEN}} \quad \nexists \; \frac{\partial F}{\partial v}(x) \; !!!$$

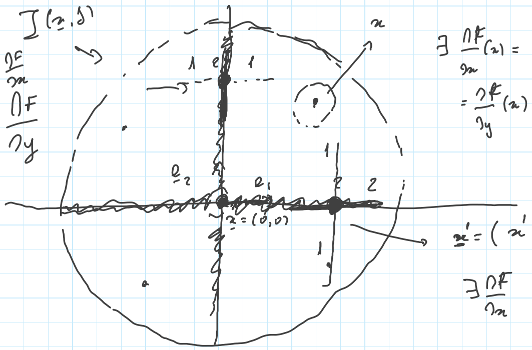


WHY THE CONDITIONS OF THE TOTAL DIFFERENTIAL THM ARE NOT SATISFIED ?

$J(x, \delta)$

$\nexists \frac{\partial F}{\partial x}$

$\exists \frac{\partial F}{\partial y}$



$\delta \in \mathbb{R}^+$

$$\exists \frac{\partial F}{\partial x}(x) = \frac{\partial F}{\partial y}(x) = 0$$

$$x' = (x', 0), \quad x' \neq 0$$

$$\exists \frac{\partial F}{\partial x}, \quad \nexists \frac{\partial F}{\partial y}$$

WHAT ABOUT

$$\exists \text{ YES } \frac{\partial F}{\partial x}(x', 0) = \lim_{t \to 0} \frac{F(x' + t e_1) - F(x')}{t} = 0 \text{ YES } \quad x' \neq 0$$

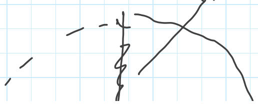
(FINITE; $=2$, $=2$)

$$? \; \exists \; \frac{\partial F}{\partial y}(x') = \lim_{t \to 0} \frac{F(x' + t e_2) - F(x')}{t} \qquad \underline{\underline{\text{NO}}} \; !!!$$

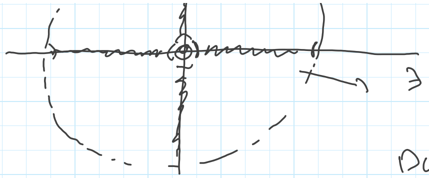
$\delta > 0$

$$\exists \frac{\partial F}{\partial y} \text{ BUT } \nexists \frac{\partial F}{\partial x}$$

$J(x, \delta)$



$x = (0, 0)$



$\exists \frac{\partial F}{\partial x}$ BUT $\nexists \frac{\partial F}{\partial y}$

DONE

BREAK

QUESTIONS ?