Wednesday, November 15, 2023 12:11 PM

NOTATIONAL IMPROVEMENT

$f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$

f DIFFERENTIABLE AT $\underline{x} \in A$.

LET $L_{\underline{x}} : \mathbb{R}^n \to \mathbb{R}$ LINEAR

IS DIFFERENTIAL

IN PLACE OF $L_{\underline{x}}$ WE WILL WRITE

LINEAR

THE POINT $\underline{x}$

$$d f (\underline{x})$$

differential

the "name" of the function

THM $f, g : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$.

LET f AND g DIFFERENTIABLE AT $\underline{x} \in A$.

THEN

1) $f+g$ IS DIFFERENTIABLE AT $\underline{x} \in A$.

MORE : $d(f+g)(\underline{x}) \overset{\text{THM}}{=} df(\underline{x}) + dg(\underline{x})$

2) fg IS DIFFERENTIABLE AT $\underline{x} \in A$

SCALAR (COEFF.) SCALAR (COEFF.)

MORE : $d(fg)(\underline{x}) = df(\underline{x}) \cdot g(\underline{x}) + f(\underline{x}) \cdot dg(\underline{x})$

LINEAR LINEAR LINEAR

3) $\lambda \in \mathbb{R}$, λf IS DIFFERENTIABLE AT $\underline{x} \in A$

MORE : $d(\lambda f)(\underline{x}) = \lambda\, df(\underline{x})$

REMEMBER IF $n = 1$

f DIFFERENTIABLE AT $\underline{x} \in A \iff f$ ADMITS DERIVATIVE $f'(x)$

MORE

$$f'(\underline{x}) = L_{\underline{x}}(1) = df(\underline{x})(1)$$

$\boxed{\text{IN } n = 1 \text{, WE INFER:}}$

FROM 1)

$$(f+g)'(\underline{x}) = d(f+g)(\underline{x})\ (1) = df(x)(1) + dg(\underline{x})(1)$$ AS LINEAR FUNCTS

$$\quad f'(\underline{x}) \qquad\qquad g'(\underline{x})$$

$$\Rightarrow (1) \Rightarrow \quad (f+g)'(\underline{x}) = f'(\underline{x}) + g'(\underline{x}) \quad \underline{\underline{\text{OK}}}$$

FROM 2)

$$(fg)'(\underline{x}) = d(fg)(\underline{x})(1) =$$

$$= \underbrace{df(\underline{x})(1)}_{f'(\underline{x})} \cdot \overset{\text{COST}}{g(\underline{x})} + \overset{\text{COST}}{f(\underline{x})}\, \underbrace{dg(\underline{x})(1)}_{g'(\underline{x})}$$

HENCE, 2) IMPLIES

$$\underline{(fg)'(\underline{x}) = f'(\underline{x}) \cdot g(\underline{x}) + f(\underline{x}) \cdot g'(\underline{x})}$$

PRODUCT FORMULA FOR DERIVATIVES (I YEAR)

$\boxed{n=1}$

3) $\Rightarrow$ $\quad \lambda \in \mathbb{R}$

$$(\lambda f)'(\underline{x}) = d(\lambda f)(\underline{x})(1) = \lambda\, df(\underline{x})(1) = \lambda f'(\underline{x})$$

$$\Rightarrow (3) \Rightarrow \quad (\lambda f)'(\underline{x}) = \lambda f'(\underline{x}).$$

PROBLEM (INFORMAL FORMULATION)

$$f: A \subseteq \mathbb{R}^n \to \mathbb{R}, \quad A \text{ OPEN}, \quad \underline{x} \in A$$

GIVEN $i = 1, 2, \dots, n$ "CONSIDER" (IF IT EXISTS)

$$\exists \; \frac{\partial f}{\partial x_i} \quad \text{OVER } A.$$

CONSIDER (IF POSSIBLE)

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial^{(2)} f}{\partial x_j \partial x_i} \quad \leftarrow$$

IF FOR $j \neq i$ MIXED DERIVATIVE OF <u>SECOND ORDER</u>

SIMILARLY, WE START FROM

$$\frac{\partial f}{\partial x_j} \quad \text{THEN}$$

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial^{(2)} f}{\partial x_i \partial x_j}$$

IS THERE A GENERAL CONNECTION BETWEEN:

$$\frac{\partial^{(2)} f}{\partial x_j \partial x_i} \quad \text{AND} \quad \frac{\partial^{(2)} f}{\partial x_i \partial x_j} \quad ???$$

EX $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 y + |y|$.

THEN $\frac{\partial f}{\partial x}(x,y) = 2xy \quad \Rightarrow$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right) = \frac{\partial}{\partial y}(2xy) = 2x$$

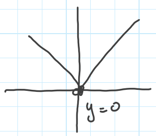
$$\| $$

$$\frac{\partial^{(2)} f}{\partial y \partial x}(x,y)$$

EXISTS EVERYWHERE ON $\mathbb{R}^2$

$h: \mathbb{R} \to \mathbb{R}$

$h(y) = |y|$



NOW, WHAT ABOUT

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y}\left(x^2 y + |y|\right)$$

THEN $\frac{\partial f}{\partial y}(x,y)$ <u>DOESN'T EXIST FOR</u> $y = 0$

$$\Downarrow$$

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right)$ <u>DOESN'T EXIST FOR $y = 0$</u>

$$\frac{\partial^{(2)} f}{\partial x \, \partial y}$$

BREAK

QUESTIONS ?

BYE BYE