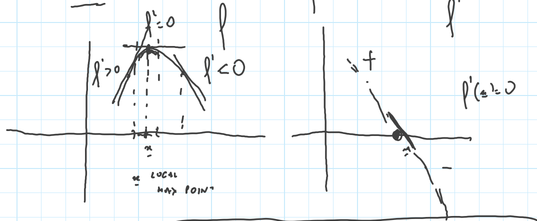
Tuesday, December 05, 2023 8:00 AM

HELLO !!!

BEGIN AT 9.10

QUESTIONS?

EX 2 $n=1$ $\quad f: A \subseteq \mathbb{R} \to \mathbb{R}$



$\Rightarrow f''(x) =$

$$\boxed{f'(f'(x)) \leq 0}$$

$$\boxed{\underline{x} \text{ LOCAL MAX} \Rightarrow f'(\underline{x}) = 0 \Rightarrow f''(x) \leq 0}$$

$f: A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$.

CONTINUITY CLASSES OF FUNCTIONS.

1) f IS OF CLASS $\mathcal{C}^{(1)}_A$ $\left(f \in \mathcal{C}^{(1)}_A\right) \overset{\text{DEF}}{\iff}$

$\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ EXIST ON A + THEY ARE EVERYWHERE CONTINUOUS ON A

NOTICE THAT

$f \in \mathcal{C}^{(1)}_A$ IMPLIES THAT TOTAL DIFF THM HOLDS! $\Rightarrow$

f DIFF OVER A.

2) f IS OF CLASS $\mathcal{C}^{(2)}_A$ $\left(f \in \mathcal{C}^{(2)}_A\right) \overset{\text{DEF}}{\iff}$

i) f IS OF CLASS $\mathcal{C}^{(1)}_A$

&

ii) ALL $\frac{\partial^{(2)} f}{\partial x_j \partial x_i}$ EXIST OVER A +

THEY ARE CONTINUOUS OVER A.

NOTICE THAT $f \in \mathcal{C}^{(2)}_A$ IMPLIES THAT SCHWARZ THM HOLDS OVER A

THAT IS

$\partial^{(2)} f \qquad \partial^{(2)} f$

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$ FOR ANY POINT OF A.

$\longrightarrow$

LOCAL MAX (MIN) POINTS FOR FUNCTS

$\rightarrow \quad f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $f \in C^{(2)}_A$.

MAIN CONSTRUCTION

FIX $\underline{x} \in A$, FIX AN (ARBITRARY) DIRECTION $v : \|v\| = 1$

DEFINE

$$r_{\underline{x},v} :]-\delta, \delta[\longrightarrow \mathbb{R}^n$$

"δ SUFFICIENTLY SMALL"

WHERE

$$r_{\underline{x},v}(t) = \underline{x} + tv \in \mathbb{R}^n \qquad t \in]-\delta, \delta[$$

THAT $r_{\underline{x},v} \equiv (r_{\underline{x},v,1}, r_{\underline{x},v,2}, \dots, r_{\underline{x},v,n})$ WHERE

$i = 1, 2, \dots, n$

$$r_{\underline{x},v,i} : t \longrightarrow x_i + t v_i \quad \Rightarrow$$

$$(r_{\underline{x},v,i})'(t) = v_i \quad . \text{ SO,}$$

FOR ANY $t \in]-\delta, \delta[$

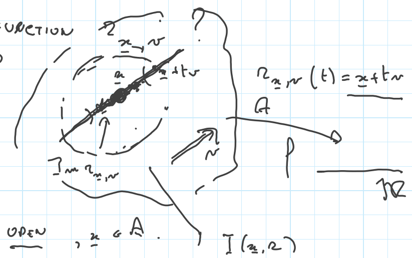
$$\exists \; \underbrace{(\underset{= v_1}{r'_{\underline{x},v,1}(t)}, \underset{= v_2}{r'_{\underline{x},v,2}(t)}, \dots, \underset{= v_n}{r'_{\underline{x},v,n}(t)}) = (v_1, \dots, v_n) = v}$$ ORIGINAL DIRECTION

WHAT IS THE FUNCTION $r_{\underline{x},v}$?

$r_{\underline{x},v}$

$-\delta \quad 0 \quad \delta$

~~]~~ $\mathbb{R}$



$f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN, $\underline{x} \in A$.

SO $\mathrm{Im}\, r_{\underline{x},v} \subseteq \mathring{A}$

, WE CAN CONSIDER THE COMPOSITION

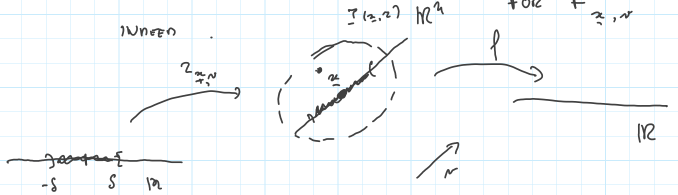
$$F_{\underline{x},v}(t) \overset{DEF}{=} (f \circ r_{\underline{x},v})(t) \underset{DEF}{=} f(r_{\underline{x},v}(t)) \qquad t \in]-\delta,\delta[$$

$$F_{\underline{x},v} : \underset{\subseteq \mathbb{R}}{]-\delta,\delta[} \longrightarrow \mathbb{R}$$

BUT NOW, WE CLAIM THAT

$\underline{x} \in A$ LOCAL MAX (MIN) FOR f $\Rightarrow$ $0 \in]-\delta,\delta[$ LOCAL MAX (MIN) FOR $F_{\underline{x},v}$

INDEED.



$\underline{x}$ MAX (MIN) FOR f $\Rightarrow$ $f(\underline{x}) \geq f(x)$ $\quad \forall x \in I(\underline{x}, r')$

$(f(\underline{x}) \leq f(x))$

$$\Longrightarrow \quad F_{\underline{x},v}(0) = f(\underline{x}) \geq F_{\underline{x},v}(t) = f(\underline{x}+tv)$$

$(\Longleftarrow)$

BUT $\underline{x}+tv \in I(\underline{x}, r')$

THEN

$\underline{x} \in A$ MAX (MIN) FOR f $\Longrightarrow$ $0 \in]-\delta,\delta[$ MAX (MIN) FOR $F_{\underline{x},v}$

$$\Rightarrow (F_{\underline{x},v})'(0) = 0 \; !!!$$

$\|$ DEF

$$(f \circ r_{\underline{x},v})'(0) \overset{\text{COMP. THM}}{=}$$

↑ DIFF

$r_{\underline{x},v}$ IS SUCH THAT $\exists\, (r'_{\underline{x},v,1}, \dots, r'_{\underline{x},v,n}) = (v_1,\dots,v_n) = v$

$$= \langle \operatorname{grad} f(\underline{x}), (r'_{\underline{x},v,1}, \dots, r'_{\underline{x},v,n}) \rangle$$

$$= \langle \operatorname{grad} f(\underline{x}), v \rangle$$

IN PLAIN WORDS, WE PROVED:

$\underline{x} \in A$ MAX (MIN) for f $\Longrightarrow$

$$\Longrightarrow \langle \operatorname{grad} f(\underline{x}), v \rangle = 0$$

TRUE FOR ANY DIRECTION $v : \|v\| = 1$.

$$\Downarrow$$

$$\text{grad} f(\underline{x}) \overset{\text{DEF}}{=} \left(\frac{\partial f}{\partial x_1}(\underline{x}), \frac{\partial f}{\partial x_2}(\underline{x}), \dots, \frac{\partial f}{\partial x_n}(\underline{x}) \right) = \underline{0} =$$

$$= (0, 0, \dots, 0)$$

THAT IS $\displaystyle \frac{\partial f}{\partial x_i}(\underline{x}) = 0 \quad \forall i = 1, 2, \dots n.$

RECALL

$$\underbrace{df}_{\text{DIFFERENTIAL}}(\underline{x}) \overset{\text{THM}}{=} \sum_{i=1}^{n} \underbrace{\frac{\partial f}{\partial x_i}(\underline{x})}_{0} \cdot dx_i = \underline{0} \in (\mathbb{R}^n)^*$$

IN PLAIN WORDS

$\underline{x}$ EA MAX (MIN) POINT $f \Longrightarrow df(\underline{x}) \equiv 0$ ($\underline{x}$ CRITICAL POINT)

IDENTICALLY ZERO FUNCT

THM $\underline{x}$ EA MAX (MIN) POINT FOR f

THEN

$\underline{x}$ IS A CRITICAL POINT

BREAK QUESTIONS?

BEGIN AGAIN A 10.17