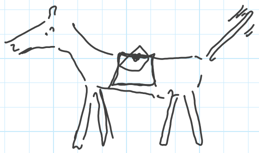
Tuesday, December 05, 2023 8:01 AM

"SADDLE POINTS"

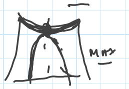


HORSE ☺

HORIZONTAL PLANE TANGENT TO THE SADDLE AT $\underline{x}$

$\underline{x}$ IS CRITICAL

MIN



MAX

$\underline{x}$ CRITICAL

$\underline{x}$ CRITICAL $\Rightarrow$

$\underline{x}$ IS NEITHER MAX NOR MIN

SO $\underline{x} \in A \subseteq \mathbb{R}^n$ CRITICAL IMPLIES

i) $\underline{x}$ MIN

ii) $\underline{x}$ MAX

iii) $\underline{x}$ SADDLE

HOW TO DECIDE ?!!

$f: A \subseteq \mathbb{R}^n \to \mathbb{R}$, A OPEN , $\underline{x} \in A$ $\qquad f \in C^{(2)}_A$

DEFINE THE MATRIX

$$H_{f(\underline{x})} = \left[\frac{\partial^{(2)} f}{\partial x_j \partial x_i}(\underline{x}) \right]_{i,j=1,2,\dots,n}$$

IT IS A SQUARE $n \times n$ MATRIX

THE HESSIAN MATRIX OF f AT $\underline{x} \in A$.

ASSUME THAT $\underline{x}$ CRITICAL

NOW, SINCE $f \in C^{(2)}_A$, THE SCHWARZ THM HOLDS $\Longrightarrow$

$f \in C^{(2)}_A$

$H_{f(\underline{x})}$ IS A SYMMETRIC MATRIX $\Rightarrow$

$H_{f(\underline{x})}$ IS A DIAGONALIZABLE (SEMISIMPLE) MATRIX (FROM LINEAR ALGEBRA)

$\Updownarrow$

THAT IS, $H_{f(\underline{x})}$ ADMITS ALL THE n EIGENVALUES.

SO, CONSIDER THE FOLLOWING COMPUTATION:

LET $v \in \mathbb{R}^n$ WE COMPUTE:

$$\langle v \times H_{f(\underline{x})} , v \rangle = ?$$

$$\left\langle (v_1, \dots, v_n) \times \left(\begin{array}{c} \vdots \\ \frac{\partial^{(2)} f(x)}{\partial x_1 \partial x_j} \\ \frac{\partial^{(2)} f(x)}{\partial x_i \partial x_j} \\ \vdots \\ \frac{\partial^{(2)} f(x)}{\partial x_n \partial x_j} \end{array} \right), (v_1, \dots, v_n) \right\rangle \overset{!}{=}$$

j-th column

$$\left\langle \left(\dots, \sum_{i=1}^{n} v_i \frac{\partial^{(2)} f(x)}{\partial x_i \partial x_j}, \dots \right), (v_1, \dots, v_j, \dots, v_n) \right\rangle =$$

j-th position

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \frac{\partial^{(2)} f(x)}{\partial x_i \partial x_j} v_i \right) v_j =$$

$$= \sum_{i,j=1}^{n} \frac{\partial^{(2)} f(x)}{\partial x_i \partial x_j} v_i v_j$$

WE PROVED:

$$\boxed{\langle v \times H_{f(x)}, v \rangle = \sum_{i,j=1}^{n} \frac{\partial^{(2)} f(x)}{\partial x_i \partial x_j} v_i v_j} \qquad (*)$$

(WE KEEP IT)

RECALL THAT x MAX (MIN) FOR $f \Rightarrow$

$\Rightarrow 0 \in]-\delta, \delta[$ IS MAX (MIN) FOR $F_{x,v} :]-\delta,\delta[\subseteq \mathbb{R} \to \mathbb{R}$ $(\dagger)$

$\Rightarrow$ BY APPLYING TWICE THE COMPOSITION THEOREM WE HAVE

$0 \in]-\delta,\delta[$ MAX (MIN) FOR $F_{x,v} \Rightarrow$

$$\Rightarrow \underbrace{F''_{x,v}(0) \leq 0}_{\text{MAX}} \quad \left(\underbrace{F''_{x,v}(0) \geq 0}_{\text{MIN}} \right)$$

$F_{x,v}(t) = f(x + tv) = (f \circ r_{x,v})(t)$ $\qquad t \in]-\delta,\delta[$

$F'_{x,v}(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x + tv) \cdot v_i$ $\qquad$ COMP THM (FIRST)

$F''_{x,v}(t) = (F'_{x,v})'(t) =$

CHAIN THM (SECOND)

$$= \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x+tv)\cdot v_i\right)(t) =$$

$$= \left(\sum_{i=1}^{n} \left\langle \operatorname{grad} \frac{\partial f}{\partial x_i}(x+tv), v \right\rangle v_i\right)(t)$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial^{(2)} f(x+tv)}{\partial x_j \partial x_i} v_j \right) \cdot v_i$$

SO, IN GENERAL

$$F''_{x,v}(t) = \sum_{i,j=1}^{n} \frac{\partial^{(2)} f(x+tv)}{\partial x_j \partial x_i} v_j v_i$$

THEN, $F''_{x,v}(0) = \sum_{i,j=1}^{n} \frac{\partial^{(2)} f(x)}{\partial x_j \partial x_i} v_j v_i$ ≤ 0 x MAX; ≥ 0 x MIN

IF x MAX

BUT COMPARE WITH $(*)$!!! ...

BUT WE HAVE

$$F''_{x,v}(0) = \sum_{i,j=1}^{n} \frac{\partial^{(2)} f(x)}{\partial x_j \partial x_i} v_j v_i = \langle v \times H_{f(x)}, v \rangle \begin{cases} \le 0 & x \text{ MAX} \\ \ge 0 & x \text{ MIN} \end{cases}$$

SO, WE HAVE PROVED.

THM x MAX (MIN) FOR $f \implies$

$$\langle v \times H_{f(x)}, v \rangle = \begin{cases} \le 0 & \text{IF } x \text{ MAX} \quad (*) \\ \ge 0 & \text{IF } x \text{ MIN} \quad (**) \end{cases}$$

$(*)$ MEANS THAT $H_{f(x)}$ IS NEGATIVELY SEMIDEFINITE

$(**)$ MEANS THAT $H_{f(x)}$ IS POSITIVELY SEMIDEFINITE

NEG. SEMIDEFINITE $\iff$ ALL THE EIGENVALUES ARE ≤ 0

POS. SEMIDEFINITE $\iff$ ALL THE EIGENVALUES ARE ≥ 0