Tuesday, December 05, 2023 8:02 AM

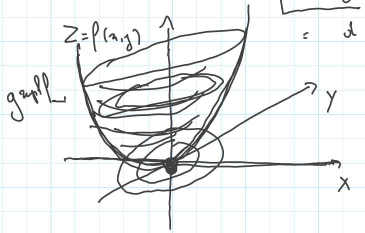
THREE SIMPLE EXAMPLES

<u>EX 1</u> $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=x^2+y^2$.

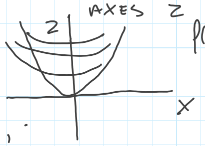
<u>PROBLEM</u>: FIND MAX/MIN POINTS FOR f.

<u>NAIVE WAY</u>. WHAT IS THE GRAPH OF f?

NOTICE THAT $z=\underline{f(x,y)}=x^2+y^2=\|(x,y)\|^2$
$= d((x,y),(0,0))^2$



graph f IS A ROTATION SURFACE IN $\mathbb{R}^3$ AROUND THE VERTICAL AXES Z



$f(x,0)=x^2$

SINCE FOR POINT $(x,0)$ WE HAVE $f(x,0)=x^2$

THERE JUST ONE MINIMUM POINT, THAT IS $\underline{x}=(0,0)$

<u>BY USING OUR THEORY</u>:

LOOK FOR CRITICAL POINT, THAT IS POINTS $(x,y)\in\mathbb{R}^2$ SUCH THAT

$$\operatorname{grad} f(x,y) = (0,0) = 0$$
$$\|$$
$$\left(\frac{\partial f}{\partial x}(x,y),\frac{\partial f}{\partial y}(x,y)\right) = (2x,2y).$$ THEN

(x,y) <u>CRITICAL</u> $\iff \operatorname{grad} f(x,y)=(0,0)=(2x,2y) \iff$

$\iff x=y=0 \implies \exists!$ CRITICAL POINT, THAT IS,

$$\underline{x}=(0,0)=\underline{0}\in\mathbb{R}^2.$$

<u>SECOND STEP</u> COMPUTE THE HESSIAN MATRIX

$$H_{f(\underline{x})}.$$

IN GENERAL, WE HAVE

$$H_{f(x,y)}=\begin{pmatrix}2&0\\0&2\end{pmatrix} \implies \underset{\underline{x}=(0,0)}{H_{f(0,0)}}=\begin{pmatrix}2&0\\0&2\end{pmatrix}$$

THIS MATRIX IS DIAGONAL

$\Downarrow$

THEN THE EIGEN VALUES

$\lambda_1=\lambda_2=2>0 \implies H_{f(\underline{x})}$ <u>POS DEFINITE</u>

$\Downarrow$ <u>THM 3</u>

$\underline{x}$ MIN !!!

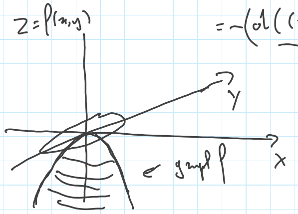
EX 2 $\quad f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = -x^2 - y^2$.

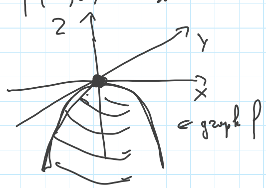
BUT $(-x^2 - y^2) = -\|(x,y)\|^2 \qquad (x,y) \in \mathbb{R}^2$

$\Downarrow$

$= -\left(\underline{d((x,y),(0,0))}\right)^2$

BUT $f(x,0) = -x^2$





SO, f HAS JUST A MAX POINT, THAT IS $\underline{x} = (0,0)$!!!

BY USING THEORY

FIRST, FIND CRITICAL POINTS THAT IS POINTS $(x,y) \in \mathbb{R}^2$ SUCH THAT

$$\operatorname{grad} f(x,y) = (0,0)$$

$$\overset{\shortparallel}{(-2x, -2y)}$$

. HENCE $(x,y) \in \mathbb{R}^2$ CRITICAL

$\iff \operatorname{grad} f(x,y) = (-2x, -2y) = (0,0) \iff$

$\iff x = y = 0 \implies \exists!$ CRITICAL POINT THAT IS $\underline{x} = (0,0)$.

COMPUTE THE HESSIAN MATRIX $H_{f(\underline{x})}$ AT $\underline{x} = (0,0)$

$$H_{f(x,y)} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

IN PARTICULAR

$$H_{f(\underline{x})} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

$\underline{x} = (0,0)$

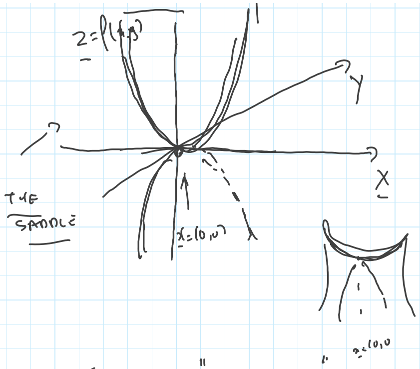
$\uparrow$ THIS IS A DIAGONA MATRIX

THEN, THE EIGENVALUES ARE

$\lambda_1 = \lambda_2 = -2 < 0 \implies H_{f(\underline{x})}$ NEGATIVELY DEFINITE

THEN, $\underline{x} = (0,0)$ IS INDEED A MAXIMUM POINT !!!

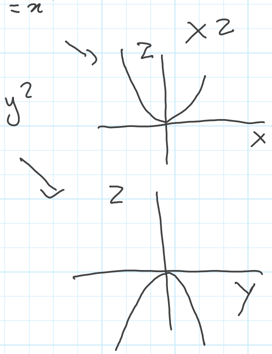
EX 3 $\quad f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 - y^2$. ???



CONSIDER

$$f(x,0) = x^2$$

$$f(0,y) = -y^2$$



SO, WE "UNDERSTOOD" THAT f HAS A UNIQUE CRITICAL POINT, SAY $\underline{x} = (0,0)$, AND THAT $\underline{x} = (0,0)$ IS A SADDLE POINT !!!

BY USING OUR THEORY

FIND CRITICAL POINTS, $(x,y) \in \mathbb{R}^2$ SUCH THAT

$$\operatorname{grad} f(x,y) = (0,0)$$

$$\parallel$$

$$(2x, -2y)$$

(x,y) CRITICAL $\iff$

$$(2x, -2y) = (0,0)$$

$$\Updownarrow$$

$$x = \; = 0$$

THEN, $\exists !$ CRITICAL POINT, NAMELY $\underline{x} = (0,0)$.

SO, CONSIDER THE HESSIAN MATRIX:

$$H_{f(x,y)} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow H_{f(\underline{x})} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

$\underline{x} = (0,0)$

THIS IS A DIAGONAL MATRIX

THEN, $H_{f(\underline{x})}$ HAS TWO EIGENVALUES: $\lambda_1 = 2 > 0$ & $\lambda_2 = -2 < 0$

SO , $Hf(\underline{x})$ IS NEITHER NEGATIVELY SEMIDEFINITE

NOR POSITIVELY SEMIDEFINITE

COR $\Longrightarrow$ $\underline{x} = (0, 0)$ IS INDEED A SADDLE POINT !!!

BYEBYE