VECTOR VALUED DIFFERENTIABLE FUNCTIONS

LET $f: A \subseteq \mathbb{R}^2 \to \mathbb{R}^n$, A OPEN, $\underline{x} \in A$.

f IS DIFFERENTIABLE AT $\underline{x} \in A$

IF AND ONLY IF

$\exists\, L_{\underline{x}} : \mathbb{R}^2 \to \mathbb{R}^n$ <u>LINEAR</u> S.T.

$$(*) \quad \lim_{h \to \underline{0} \in \mathbb{R}^2} \frac{f(\underline{x}+h) - f(\underline{x}) - L_{\underline{x}}(h)}{\|h\|_2} = \underline{0} \in \mathbb{R}^n$$

$f \equiv (f_1, f_2, \dots, f_n)$

$L_{\underline{x}} \equiv (L_{\underline{x},1}, \dots, L_{\underline{x},n})$

$(*)$ IS EQUIVALENT

$$(**) \quad \begin{cases} \lim\limits_{h \to \underline{0} \in \mathbb{R}^2} \dfrac{f_1(\underline{x}+h) - f_1(\underline{x}) - L_{\underline{x},1}(h)}{\|h\|_2} = 0 \in \mathbb{R} \\ \lim\limits_{h \to \underline{0} \in \mathbb{R}^2} \dfrac{f_2(\underline{x}+h) - f_2(\underline{x}) - L_{\underline{x},2}(h)}{\|h\|_2} = 0 \in \mathbb{R} \\ \text{---} \\ \lim\limits_{h \to \underline{0} \in \mathbb{R}^2} \dfrac{f_n(\underline{x}+h) - f_n(\underline{x}) - L_{\underline{x},n}(h)}{\|h\|_2} = 0 \in \mathbb{R} . \end{cases}$$

$(\dagger)$ THE i-th SCALAR COMP $L_{\underline{x},i}$ OF $L_{\underline{x}} : \mathbb{R}^2 \to \mathbb{R}^n$ IS PRECISELY THE DIFFERENTIAL $df_i(\underline{x})$ OF THE SCALAR COMPONENT f_i OF $f: A \subseteq \mathbb{R}^2 \to \mathbb{R}^n$

$(\dagger\dagger)$ IN PLAIN NOTATION:

$$L_{\underline{x},i} = df_i(\underline{x}) \qquad i = 1, 2, \dots, n$$

!!!

<u>NOW</u>, WE FIX CANONICAL BASES BOTH IN $\mathbb{R}^2$ AND $\mathbb{R}^n$

WE CAN DESCRIBE $L_{\underline{x}}(h)$, $h \in \mathbb{R}^2$

AS A PRODUCT OF MATRICES $\qquad h = (h_1, \dots, h_2) \in \mathbb{R}^2$

$$\underbrace{\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}}_{M_{L_{\underline{x}}}} \times \underbrace{\begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_2 \end{pmatrix}}_{h^t} = \begin{pmatrix} L_{\underline{x},1}(h) \\ L_{\underline{x},2}(h) \\ \vdots \\ L_{\underline{x},n}(h) \end{pmatrix}$$

$M_{L_{\underline{x}}}$ MATRIX OF ORDER $n \times 2$

$\|$ BY $(\dagger\dagger)$

WHAT IS THIS MATRIX ???

$$\begin{pmatrix} df_1(\underline{x})(h) \\ df_2(\underline{x})(h) \end{pmatrix}$$

$$\begin{pmatrix} df_2(x)(h) \\ \vdots \\ df_n(x)(h) \end{pmatrix} \leftarrow$$

$$= \begin{pmatrix} \langle \mathrm{grad} f_1(x), h \rangle \\ \langle \mathrm{grad} f_2(x), h \rangle \\ \vdots \\ \langle \mathrm{grad} f_n(x), h \rangle \end{pmatrix}$$

$$\| \quad \begin{pmatrix} \sum_{j=1}^{2} \frac{\partial f_1(x)}{\partial x_j} h_j \\ \sum_{j=1}^{2} \frac{\partial f_2(x)}{\partial x_j} h_j \\ \dots \\ \sum_{j=1}^{2} \frac{\partial f_n(x)}{\partial x_j} h_j \end{pmatrix}$$

OUR PROBLEM IS

$n \times 2$ MATRIX $\quad M_{L_3}$

$$\begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \dots & \frac{\partial f_1(x)}{\partial x_2} \\ \vdots & & \vdots \\ \frac{\partial f_n(x)}{\partial x_1} & \dots & \frac{\partial f_n(x)}{\partial x_2} \end{pmatrix} \times \begin{pmatrix} h_1 \\ \vdots \\ h_2 \end{pmatrix} \overset{!}{=} \begin{pmatrix} \sum_{j=1}^{2} \frac{\partial f_1(x)}{\partial x_j} h_j \\ \vdots \\ \sum_{j=1}^{2} \frac{\partial f_n(x)}{\partial x_j} h_j \end{pmatrix}$$

THIS IS THE JACOBIAN MATRIX

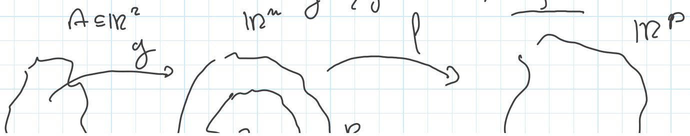
$$J f(x) \overset{!}{=} \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_2}(x) \\ \vdots & & \vdots \\ \frac{\partial f_n(x)}{\partial x_1} & \dots & \frac{\partial f_n(x)}{\partial x_2} \end{pmatrix} \quad !!!$$

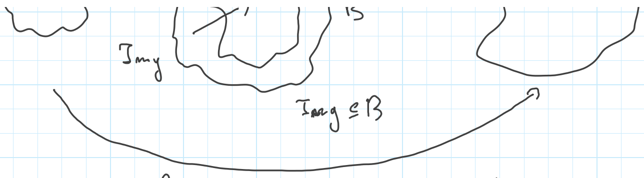
THM (GENERAL FORM OF THE COMPOSITION THM)

$g : A \subseteq \mathbb{R}^2 \to \mathbb{R}^n \quad , f : B \subseteq \mathbb{R}^n \to \mathbb{R}^p$

A, B OPEN.

ASSUME THAT $\mathrm{Im}\, g = \{ g(x) \in \mathbb{R}^n ; x \in A \} \subseteq B$.





$$(f \circ g)(x) = f(g(x)) \qquad \forall x \in A.$$

NOW $a \in A$, $b = g(a) \in B$

<u>ASSUME</u> 1) g DIFFERENTIABLE AT $a \in A$

2) f DIFFERENTIABLE AT $b = g(a) \in B$

NOW, THE FUNCTION

$$f \circ g : A \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^p$$

IS <u>DIFFERENTIABLE</u> AT $a \in A$.

BUT, FURTHERMORE

$$d(f \circ g)(a) = \underbrace{df(b)}_{\text{LINEAR}} \circ \underbrace{dg(a)}_{\text{LINEAR}} \qquad (*)$$

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[rr, "d(f\circ g)(a)\ \text{LINEAR}"] \arrow[dr, "dg(a)\ \text{LINEAR}"'] & & \mathbb{R}^p \\
& \mathbb{R}^m \arrow[ur, "df(b)\ \text{LINEAR}"'] &
\end{tikzcd}$$

BUT, MORE EXPLICITELY $(*)$ IN MATRIX FORM BECOMES

$$\begin{tikzcd}
d(f\circ g)(a) \arrow[d] & df(b) \arrow[d] & dg(a) \arrow[d] \\
J_{(f\circ g)(a)} & J_{f(b)} & J_{g(a)}
\end{tikzcd}$$

ROW-COLUMN PRODUCT.

$$J_{(f\circ g)(a)} = J_{f(b)} \times J_{g(a)} \qquad \boxed{\text{OK}}$$

GIVEN

$$p \times 2 \quad = \quad p \times m \qquad m \times 2$$

THE GENERAL FORM OF DINI'S THM

THE IMPLICIT FUNCTION THM

LET $F: A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^{n-2}$, A OPEN, $\underline{\underline{F \in C^{(1)}_A}}$

CONSIDER $Z_F = \{ (x_1, \dots, x_n) \in \mathbb{R}^n \; ; \; F(x_1, \dots, x_n) = \underline{0} \in \mathbb{R}^{n-2} \}$

LET $y = (y_1, \dots, y_n) \in Z_F$ THAT IS $F(y_1, \dots, y_n) = (0, \dots, 0) \in \mathbb{R}^{n-2}$

(HP) ASSUME THAT

$$rk \, J_{F(y)} \quad \text{IS MAXIMAL} \left(\text{i.e. } rk \, J_{F(y)} \overset{(*)}{=} n-2 \right)$$

n COLUMNS

$$(*) \iff J_{F(y)} \quad \begin{array}{c c} & \begin{array}{ccccc} j_1 & j_2 & & & j_{n-2} \end{array} \\ \begin{array}{c} n-2 \\ \text{ROW} \end{array} & \left(\begin{array}{ccccc} | & | & \cdots & | & \\ | & | & \cdots & | & \end{array} \right) \end{array}$$

WE CONSIDER THE SQUARE $(n-2) \times (n-2)$ SUBMATRIX OF $J_{F(y)}$

$\Downarrow$

HAS DETERMINANT

$\neq 0$!!!

BREAK QUESTIONS?

BEGIN AT 10.16