Monday, December 11, 2023 11:37 AM

REGULAR (DIFF.) VARIETY IN $\mathbb{R}$

$V \subseteq \mathbb{R}^n$ IS A REGULAR VARIETY

OF DIMENSION 2 AND CLASS $C^{(k)}$, $k \geq 1$

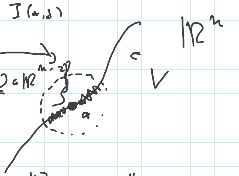
IF AND ONLY IF

$\forall\, \alpha \in V \quad \exists\, I(\alpha, \delta) \subseteq \mathbb{R}^n \qquad \delta \in \mathbb{R}^+$

AND

$f : I(\alpha, \delta) \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^{n-2}$, $\quad f \in C^{(k)}_{I(\alpha,\delta)}$, $k \geq 1$

SUCH THAT :



i) $V \cap I(\alpha, \delta) = \{ x \in I(\alpha,\delta) : f(x) = \underline{0} \in \mathbb{R}^{n-2} \}$

$\Updownarrow$

IN SCALAR COMPONENTS

SYSTEM OF $n-2$ EQS INTO n INDETERMINATES $\rightarrow$

$$\begin{cases} f_1(x_1, x_2, \dots, x_n) = 0 \in \mathbb{R} \\ f_2(x_1, x_2, \dots, x_n) = 0 \in \mathbb{R} \\ \text{— — —} \\ f_{n-2}(x_1, x_2, \dots, x_n) = 0 \in \mathbb{R} \end{cases}$$

$\swarrow$ "CARTESIAN DESCRIPTION"

ii) (REGULARITY !!!) $\qquad \forall\, \alpha \in V$

$rk\left(J_{f(\alpha)}\right)$ IS MAXIMAL , THAT IS

$$rk\left(J_{f(\alpha)}\right) = n - 2 .$$

WHY ii) IS "REGULARITY" ? BUT ,

$f \in C^{(k)}$, $k \geq 1$

$+$

$rk\left(J_{f(\alpha)}\right)$ MAXIMAL $\qquad \alpha \in Z_f$

ARE PRECISELY THE HPS OF THE IMPLICIT FUNCTION THM !!

THEN , TO SAY THAT $V \subseteq \mathbb{R}^n$ IS

A REGULAR VARIETY MEANS THAT

THE IMPLICIT FUNCT THM APPLIES

FOR EVERY POINT $\cancel{\alpha} \in V$!!!

LOCALLY SPEAKING, THE POINT OF V

CAN BE REGARDED AS POINTS WHOSE COORDINATES ARE

i) $n-r$ DEPENDENT (AS FUNCTIONS) FROM

ii) r INDEPENDENT COORDINATES. } $(*)$ → THE DIMENSION OF V IS r

+ IMPL. FUNCT. THEOREM

THE FUNCTS

DEPENDENT COORD.

$$y_1 = \varphi_1(x_1, \dots, x_r) \in C^{(k)}, \quad k \geq 1$$

$$y_2 = \varphi_2(x_1, \dots, x_r) \in C^{(k)}, \quad k \geq 1$$

$$\vdots$$

$$y_{n-r} = \varphi_{n-r}(x_1, \dots, x_r) \in C^{(k)}, \quad k \geq 1$$

$$V = \{(x,y,z) \in \mathbb{R}^3 ;\ x^2+y^2+z^2 = 4,\ z = 1\} \subseteq \mathbb{R}^3$$

IS V A REGULAR VARIETY ???

LET $f: \mathbb{R}^3 \to \mathbb{R}^2$, $f = (f_1, f_2)$,

WHERE

$$f_1(x,y,z) = x^2+y^2+z^2-4, \quad f_2(x,y,z) = z-1.$$

THEN $V = \{(x,y,z) \in \mathbb{R}^3 ;\ f_1(x,y,z) = 0, f_2(x,y,z) = 0\}$

SO i) DONE

WHAT ABOUT COND ii) ??

$$J_f(x,y,z) = \begin{pmatrix} 2x & 2y & 2z \\ 0 & 0 & 1 \end{pmatrix}. \qquad (x,y,z) \in \mathbb{R}^3$$

UNDER WHICH CONDS $\operatorname{rk} J_f(x,y,z)$ IS NOT MAXIMAL.

$\iff x = y = 0$ BUT THIS IMPLIES FOR POINTS

$$(0, 0, z) \in \mathbb{R}^3$$

BUT CAN THESE POINTS BELONG TO V?

IN ORDER TO SATISFY $f_1(x,y,z) \iff z = \pm 2$

" " " " $f_2(x,y,z) \iff z = 1$

CONTRADICTION !

THEREFORE IF $(x,y,z) \in \mathbb{R}^3$ IS SUCH THAT

$$\operatorname{rk} J_{f(x,y,z)} \ \underline{\text{NOT MAXIMAL}} \iff (x,y,z) \notin V$$

$$\Updownarrow$$

$$\underline{\forall \alpha \in V} \quad \operatorname{rk}\left(J_{f(\alpha)}\right) = 2 \quad \underline{\text{MAXIMAL}}.$$

FIX $\alpha \in V$ $\quad$ ~~REGULAR VARIETY IN $\mathbb{R}^n$~~

<u>BREAK</u> QUES