Monday, December 11, 2023 11:38 AM

Curves in $\mathbb{R}^n$

A **curve** in $\mathbb{R}^n$ is a map

$$\varphi : [a,b] \subseteq \mathbb{R} \to \mathbb{R}^n, \qquad \varphi \equiv (\varphi_1, \varphi_2, \dots, \varphi_n)$$

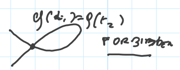
where $\varphi_1, \varphi_2, \dots, \varphi_n \in C^{(1)}_{[a,b]}$

The support of φ is

$$\operatorname{supp} \varphi = \{ \varphi(t) \in \mathbb{R}^n ; \; t \in [a,b] \}$$

A curve $\varphi : [a,b] \to \mathbb{R}^n$ is

simple open $\overset{\text{DEF}}{\Longleftrightarrow}$



$$\varphi : [a,b] \xrightarrow[1-1]{\text{onto}} \operatorname{supp} \varphi$$

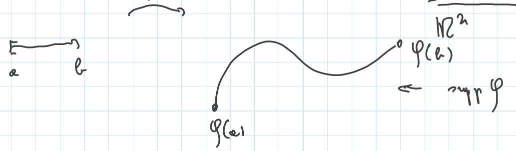
$+$

φ is an **homeomorphism** between $[a,b]$ and $\operatorname{supp} \varphi$

that is

φ is continuous $+$ $\varphi^{-1} : \operatorname{supp} \varphi \to [a,b]$ is **also continuous**

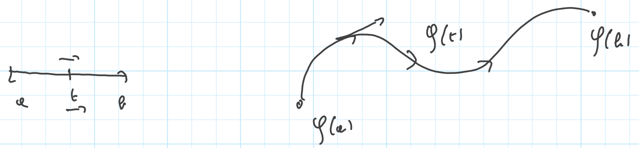
φ homeomorphism



$+$

φ is **regular** $\Longleftrightarrow$ $\forall\, t \in \,]a,b[$

$$\varphi'(t) = (\varphi_1'(t), \varphi_2'(t), \dots, \varphi_n'(t)) \neq \underline{0} \in \mathbb{R}^n.$$



DEF $h \in \mathbb{R}^n$ tangent to the variety V at the point $\alpha \in V$

IF AND ONLY IF

$\exists\ \varphi : [a,b] \to \mathbb{R}^n$ SIMPLE OPEN + REGULAR CURVE
s.t.

i) $\operatorname{supp} \varphi \subseteq V$

ii) $\varphi(0) = a$

iii) $h = (\varphi_1'(0), \varphi_2'(0), \dots, \varphi_n'(0)) \neq \underline{0}$

NOW, SET

$\to T(a) = \{ h \in \mathbb{R}^n ;\ h \text{ TG TO } V \text{ at } a \} \cup \{\underline{0}\} \subseteq \mathbb{R}^n$

IT IS THE <u>TANGENT SET</u>.

SO, LET V A REGULAR VARIETY OF DIM r IN $\mathbb{R}^n$

<u>FOR $a \in V$</u>, AND LET $f : I(a,\delta) \subseteq \mathbb{R}^n \to \mathbb{R}^{n-r}$ THAT <u>PROVIDES EQS</u>.
$f \in C^{(k)},\ k \geq 1$

<u>THM 1</u> $\boxed{T(a) \overset{\text{THM}}{=} \operatorname{Ker} df(a)}$,

WHERE $\operatorname{Ker} df(a) \overset{\text{DEF}}{=} \{ h \in \mathbb{R}^n ;\ df(a)(h) = \underline{0} \in \mathbb{R}^{n-r} \}$

$(df(a) : \mathbb{R}^n \to \mathbb{R}^{n-r}$ LINEAR$)$

SINCE $df(a)$ IS <u>LINEAR</u> $\Rightarrow$ $\operatorname{Ker} df(a)$ IS A VECTOR <u>SUBSPACE OF $\mathbb{R}^n$</u>

$T(a)$
$\uparrow$
<u>TANGENT SPACE</u>

WHAT IS THE DIMENSION OF

$T(a) \overset{\text{THM}}{=} \operatorname{Ker} df(a)$???

RECALL <u>GRASSMANN LAWS</u>! (1844)

$$\underline{\dim(\operatorname{Ker} df(a)) + \dim(\operatorname{Im} df(a)) = n}$$!!...

BUT, IN OUR SITUATION, WHAT IS $\dim(\operatorname{Im} df(a))$???

RECALL THAT

$J_{f(a)}$ IS THE MATRIX THAT REPRESENTS $df(a)$

$\Downarrow$

$\dim(\operatorname{Im} df(a)) = \underline{\text{column rk}}\,(J_{f(a)}) = \operatorname{rk}(J_{f(a)}) = n-r$ BY <u>REG. HP</u>

SO $\dim\left(\operatorname{Ker} df(\alpha)\right) + (n-r) = n$

$\Rightarrow \dim\left(\operatorname{Ker} df(\alpha)\right) = n-(n-r) = r.$

THM 2 (IN MATRIX LANGUAGE)

$$T(\alpha) = \left\{ h \in \mathbb{R}^n ; \ J_{f(\alpha)} \times \underbrace{\begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}}_{h^t} = \underbrace{\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}}_{0^t \in \mathbb{R}^{n-r}} \right\}$$

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\alpha) & \cdots & \frac{\partial f_1}{\partial x_n}(\alpha) \\ - & - & - \\ \frac{\partial f_{n-r}}{\partial x_1}(\alpha) & \cdots & \frac{\partial f_{n-r}}{\partial x_n}(\alpha) \end{pmatrix} \times \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\Downarrow$ INDETERMINATES

$$\begin{cases} \frac{\partial f_1}{\partial x_1}(\alpha)\cdot h_1 + \cdots + \frac{\partial f_1}{\partial x_n}(\alpha)\cdot h_n = 0 \\ \quad - \quad - \quad - \\ \frac{\partial f_{n-r}}{\partial x_1}(\alpha)\cdot h_1 + \cdots + \frac{\partial f_{n-r}}{\partial x_n}(\alpha)\, h_n = 0 \end{cases}$$

A HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

THM 3 $T(\alpha) = \left\{ h \in \mathbb{R}^n ; \ \langle \operatorname{grad} f_i(\alpha), h \rangle = 0 \ \ \forall i = 1, 2, \dots, n \right\}$

LET $N(\alpha) = \left(T(\alpha)\right)^{\perp}$ ORTHOGONAL COMPL.

NORMAL SPACE

$\quad = \left\{ v \in \mathbb{R}^n ; \ \langle v, h \rangle = 0 \ \ \forall h \in T(\alpha) \right\}$

SINCE $\dim T(\alpha) = r \Rightarrow \dim N(\alpha) = n - r$

THM 3 IMPLIES (TRIVIALLY) THM

$\operatorname{grad} f_1(\alpha), \dots \operatorname{grad} f_{n-r}(\alpha) \in N(\alpha)$!!!

THE ROWS OF $Jf(\alpha)$

BUT, BY HP : $\operatorname{rk}\left(J_{f(\alpha)}\right) = n - r$

WE HAVE $\operatorname{grad} f_1(\alpha), \dots, \operatorname{grad} f_{n-r}(\alpha)$ ARE LINEARLY INDEPENDENT !!!.

THM THE SET

$\quad \ldots f(\alpha) \qquad \ldots \operatorname{grad} f \ldots (\alpha)$

$\operatorname{grad} f_1(\alpha), \ldots, \operatorname{grad} f_{m-k}(\alpha)$

IS BASIS

OF THE NORMAL SPACE $N(\alpha)$. !!!