MEASURE THEORY AND LEBESGUE INTEGRATION

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Contents

1 Outer measure and Lebesgue measure in $\mathbb{R}^n$

1.1 Preliminary remarks. On the cardinality of infinite sets. Countable sets

Definition 1.

- *Given two sets X, Y, (finite or infinite), we say that they are EQUICARDINAL (0R, THEY HAVE THE SAME CARDINALITY) if and only if there exits a BIJECTION $f : X \to Y$.*
- *A set X is said to be COUNTABLE if and only if it is equicardinal to the set $\mathbb{N}$ (the set of NATURAL NUMBERS.*

Remark 1. Cantor Theorems

- *The union of a finite or countable family of finite or countable sets is a finite or countable set.*
- *The cartesian product of finite or countable sets is a finite or countable set.*
- *The sets $\mathbb{Z}^+$ (positive integers), $\mathbb{Z}$ (relative integers), $\mathbb{Q}$ (rational numbers) are COUNTABLE.*
- *Given a set X, let $mathbf P(X) = \{A; A \subseteq X\}$. The set $\mathbf{P}(X)$ has cardinality strictly greater than the cardinality of X, that is, there exist injective functions $f : X \hookrightarrow \mathbf{P}(X)$, but NOT VICEVERSA.*

- *The sets that are equicardinal to $\mathbb{R}$ (real numbers) have continuous cardinality, that is strictly greater than the countable cardinality.*
- *The set $\mathbb{R} - \mathbb{Q}$ (irrational numbers) has continuous cardinality.*

1.2 Lebesgue coverings

Let $a_j < b_j \in \mathbb{R}$, $j = 1, \ldots, n$; the set

$$I = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n; a_j < x_j < b_j,\ j = 1, \ldots, n\}$$

is called *LIMITED OPEN INTERVAL* in $\mathbb{R}^n$.

The (positive) real number

$$\mu(I) = \Pi_{j=1}^{n}\ (b_j - a_j)$$

is the *measure* of the interval I.

Let $A \subseteq \mathbb{R}^n$, and let $\{I_k; k \in \mathcal{A}\}$ be a family of limited open intervals in $\mathbb{R}^n$, with $\mathcal{A}$ FINITE or AT MOST COUNTABLE; the family $\{I_k; k \in \mathcal{A}\}$ is said to be a *LEBESGUE COVERING* of A if and only if

$$A \subseteq \bigcup_{k \in \mathcal{A}} I_k.$$

In the following, we will denote by the symbol $\mathcal{I}_A$ the set of all LEBESGUE COVERINGS of the set $A \subseteq \mathbb{R}^n$.

1.3 Outer measure in $\mathbb{R}^n$

The OUTER MEASURE of the set $A \subseteq \mathbb{R}^n$ is the "extended real number" (that is, that belongs $\mathbb{R} \cup \{\infty\}$) defined as follows:

$$\mu^*(A) = inf\ \{\sum_{k \in \mathcal{A}} \mu(I_k);\ \{I_k; k \in \mathcal{A}\} \in \mathcal{I}_A\}.$$

If $\mu^*(A) = \infty$, we say that A has INFINITE outer measure; otherwise, we say that A has FINITE outer measure.

Example 1. *The outer measure of any singleton set equals zero. In symbols, for every $x \in \mathbb{R}^n$, we have $\mu^*(\{x\}) = 0$.*

Indeed, let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Given any $\varepsilon \in \mathbb{R}^+$, set

$$I(x; \varepsilon) = \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n; x_j - \frac{\varepsilon}{2} < y_j < x_j + \frac{\varepsilon}{2}, j = 1, \ldots, n\}.$$

Since $I(x;\varepsilon)$ is a limited open interval in $\mathbb{R}^n$ e $x \in I(x;\varepsilon)$, the singleton set $\{I(x;\varepsilon)\}$ is a Lebesgue covering of the (singleton) set $\{x\}$. Since $\mu(I(x;\varepsilon)) = \varepsilon^n$, we have $\mu^*(\{x\}) \leq \varepsilon^n$; hence, since $\varepsilon \in \mathbb{R}^+$ is arbitrary, it follows that $\mu^*(\{x\}) = 0$.

The following fundamental result implies that, *in the case of limited open intervals* I the outer measure $\mu^*(I)$ coincides with the measure $\mu(I)$ given by definition, and, furthermore that the outer measure of I equals the outer measure of the *closure* $\overline{I}$ (limited closed interval) of I.

Theorem 1. *Let I be a limited open interval in $\mathbb{R}^n$. Then*

$$\mu^*(I) = \mu(I) = \mu^*(\overline{I}).$$

Since $A_1 \subseteq A_2 \subseteq \mathbb{R}^n$ implies $\mathcal{I}_{A_2} \subseteq \mathcal{I}_{A_1}$, we have:

Theorem 2. *(monotonicity of the outer measure)*

If$A_1 \subseteq A_2 \subseteq \mathbb{R}^n$, then $\mu^*(A_1) \leq \mu^*(A_2)$.

Corollary 1. $\mu^*(\emptyset) = 0$.

Example 2. *The real line $\mathbb{R}$ has infinite outer measure. Indeed, for every $n \in \mathbf{Z}^+$, the limited open interval $]-n,n[$ has outer measure $\mu^*(]-n,n[) = \mu(]-n,n[) = 2n$ and, furthermore $]-n,n[\subseteq \mathbb{R}$; then $\mu^*(\mathbb{R}) \geq 2n$, for every $n \in \mathbf{Z}^+$. It follows that $\mu^*(\mathbb{R}) = \infty$.*

Theorem 3. *(countable subadditivity)*

Let $\{A_k; k \in \mathcal{A}\}$ be an at most countable family of subsets of $\mathbb{R}^n$. Then:

$$\mu^*(\bigcup_{k\in\mathcal{A}} A_k) \leq \sum_{k\in\mathcal{A}} \mu^*(A_k).$$

PROOF. The statement is trivially true whenever $\sum_{k\in\mathcal{A}} \mu^*(A_k) = +\infty$.

Let us consider the case $\sum_{k\in\mathcal{A}} \mu^*(A_k) =< \infty$.

From the definition of outer measure, it follows that, for every fixed-arbitrary $\varepsilon \in \mathbb{R}^+$, for every i $k \in \mathcal{A}$ THERE EXISTS a Lebesgue covering $\{I_{k_j}; j \in \mathcal{A}_k\}$ of A_k such that

$$\sum_{j\in\mathcal{A}_k} \mu(I_{k_j}) < \mu^*(A_k) + \frac{\varepsilon}{2^k}.$$

The set

$$\{I_{k_j}; j \in \mathcal{A}_k\}, k \in \mathcal{A}\}$$

is a Lebesgue covering of $\bigcup_{k\in\mathcal{A}} A_k$). Hence

$$\mu^*(\bigcup_{k\in\mathcal{A}} A_k) \le \sum_{k\in\mathcal{A}} \sum_{j\in\mathcal{A}_k} \mu(I_{k_j}) < \sum_{k\in\mathcal{A}} (\mu^*(A_k) + \frac{\varepsilon}{2^k}) \le \sum_{k\in\mathcal{A}} \mu^*(A_k) + \varepsilon.$$

Since $\varepsilon \in \mathbb{R}^+$ is arbitrary, it follows

$$\mu^*(\bigcup_{k\in\mathcal{A}} A_k) \le \sum_{k\in\mathcal{A}} \mu^*(A_k).$$

Corollary 2. *let A be a countable subset of $\mathbb{R}^n$. Then $\mu^*(A) = 0$.*

Indeed, writing $A = \bigcup_{a\in A} \{a\}$, the set A turns out to be a countable union of singletons; hence,

$$\mu^*(A) \le \sum_{a\in A} \mu^*(\{a\}) = 0.$$

Example 3. *Let $\mathbb{Q}$ be the subset of rational numbers. Then $\mu^*(\mathbb{Q}) = 0$.*

Furthermore, we have

Theorem 4. *let $A, B \subseteq \mathbb{R}^n$ such that $d(A,B) =^{def} \; inf\{d(a,b); a \in A, b \in B\} > 0$. Then $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.*

Remark 2.

- *Notice that the condition $d(A,B) > 0$ implies that $A \cap B = \emptyset$, but NON VICEVERSA. (for example $]0,1],]1,2]$ are disjoint intervals in $\mathbb{R}$, but $d(]0,1],]1,2]) = 0$.)*
- *The preceding assertion is FALSE if we replace the condition $d(A,B) > 0$ is replaced by the weaker condition $A \cap B = \emptyset$. In other words, there exist pairs of disjoint subsets A , B such that*

$$\mu^*(A \cup B) < \mu^*(A) + \mu^*(B).$$

1.4 Measurable subsets in $\mathbb{R}^n$

Definition 2. *(after C. Caratheodory) A subset A of $\mathbb{R}^n$ is said to be MEASURABLE if and only if, for every $E \subseteq \mathbb{R}^n$, we have:*

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c_{\mathbb{R}^n}),$$

where $A^c_{\mathbb{R}^n} = \mathbb{R}^n - A$ denotes the complementary subset of A in $\mathbb{R}^n$.

Remark 3.

- *The inequality*
$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c_{\mathbb{R}^n})$$
is always true, by subadditivity.
- *There existsubsets $\mathbb{R}^n$ that are NON MEASURABLE.*
- *In general, if A_1 e A_2 are NON MEASURABLE DISJOINT subsets in in $\mathbb{R}^n$, it may be FALSE that*
$$\mu^*(A_1 \cup A_2) = \mu^*(A_1) + \mu^*(A_2).$$
- *From the simmetry of the definition, it immediately follws that A is measurable if and only if its complementary subset $A^c_{\mathbb{R}^n}$ is measurable.*

In the case of measurable subsets, their *MEASURE* is, by definition, their outer measure and we will write $\mu(A)$ in place of $\mu^*(A)$.

1.5 Fundamental properties of the measure and of measurable sets

Theorem 5. *If A_1, A_2 are measurable subsets in $\mathbb{R}^n$, then*
$$A_1 \cup A_2,\ A_1 \cap A_2,\ A_1 - A_2,\ A_2 - A_1$$
are measurable subsets.

Theorem 6. *Let $\{A_k; k \in \mathcal{A}\}$ be an at most countable family of measurable subsets in $\mathbb{R}^n$. Then $\bigcup_{k\in\mathcal{A}} A_k$ is measurable.*

Theorem 7. *(Countable additivity for disjoint measurable sets)*

Let $\{A_k; k \in \mathcal{A}\}$ be an at most countable family of MUTUALLY DISJOINT measurable subsets in $\mathbb{R}^n$. Then:
$$\mu(\bigcup_{k\in\mathcal{A}} A_k) = \sum_{k\in\mathcal{A}} \mu(A_k).$$

Example 4.

- *Let $A \subseteq \mathbb{R}^n$ such that $\mu^*(A) = 0$. Then A is measurable.*

Indeed, for every $E \subseteq \mathbb{R}^n$, we have:
$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c_{\mathbb{R}^n}),$$
by subadditivity. On the hand, we have:
$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c_{\mathbb{R}^n}) = 0 + \mu^*(E \cap A^c_{\mathbb{R}^n}),$$
by monotonicity.

- *Let $A \subseteq \mathbb{R}^n$, A countable. Then A is measurable and $\mu(A) = 0$.*

  Indeed, by countable subadditivity, we have :

$$\mu^*(\bigcup_{a \in A} \{a\}) \leq \sum_{a \in A} \mu^*(\{a\}) = 0.$$

- *Consider the set $[0,1] - \mathbb{Q} \subset \mathbb{R}$ (the subset of irrational numbers that belong to the closed interval $[0,1]$).*

  The set $[0,1] - \mathbb{Q}$ is measurable? If YES, compute its measure?

  Notice that $[0,1]$ is measurable since it is closed (see Theorem **??** *below), and, then, also $[0,1] - \mathbb{Q}$ is measurable (WHY?). Furthermore, we have*

$$\mu([0,1]) = 1 = \mu([0,1] - \mathbb{Q}) + \mu([0,1] \cap \mathbb{Q}) = \mu([0,1] - \mathbb{Q}) + 0,$$

  and, hence, $\mu([0,1] - \mathbb{Q}) = 1$.

1.6 Limit theorems for "nested" sequences of measurable sets

Proposition 1. *Let $(A_k)_{k \in \mathbb{N}}$ be a sequence of measurable subsets in $\mathbb{R}^n$ such that $A_k \subseteq A_{k+1}$, for every $k \in \mathbb{N}$. The*

$$\mu(\bigcup_{k=0}^{\infty} A_k) = lim_{k \to \infty} \mu(A_k).$$

Proof. The countable union $\bigcup_{k=0}^{\infty} A_k$ is measurable.

Set $B_0 = A_0$, $B_{k+1} = A_{k+1} - A_k$, for every $k > 0$.

The set B_k are measurable, mutually disjoint and $\bigcup_{k=0}^{\infty} B_k = \bigcup_{k=0}^{\infty} A_k$. Then

$$\mu(\bigcup_{k=0}^{\infty} A_k) = \mu(\bigcup_{k=0}^{\infty} B_k) = \sum_{k=0}^{\infty} \mu(B_k) =^{def} lim_{k \to \infty} (\sum_{j=0}^{k} \mu(B_j);$$

on the other hand

$$\sum_{j=0}^{k} \mu(B_j) = \mu(\bigcup_{j=0}^{k} B_j) = \mu(\bigcup_{j=0}^{k} A_j) = \mu(A_k).$$

□

Example 5. *Given a constant* $\theta \in \mathbb{R}$*, consider the line* $A = \{(x, y) \in \mathbb{R}^2; y = \theta\}$.

The set A *is measurable, since it is closed. For every* $k \in \mathbb{Z}^+$*, let* $A_k = \{(x, y) \in \mathbb{R}^2; k < x < k, y = \theta\} \subset A$.

Each A_k *is measurable, with zero measure and* $\bigcup_{k=0}^{\infty} A_k = A$*. from the preceding result, it follows that* A *has zero measure in* $\mathbb{R}^2$.

Proposition 2. *Let* $(A_k)_{k \in \mathbb{N}}$ *be a sequence of measurable subsets in* $\mathbb{R}^n$ *such that* $A_k \supseteq A_{k+1}$*, such that* $k \in \mathbb{N}$*. Assume the further condition*

$$\exists k \in \mathbb{N} \text{ tale che } \mu(A_k) < \infty \qquad (\dagger)$$

Then

$$\mu(\bigcap_{k=0}^{\infty} A_k) = lim_{k \to \infty} \mu(A_k).$$

Remark 4. *If we delete condition* $(\dagger)$*, the assertion is FALSE. For example, let us consider the sequence of "half-lines"* $A_k =]k, +\infty[, k \in \mathbb{N}$*. Clearly* $\mu(]k, +\infty[) = +\infty, \forall k$*. On the other hand,* $\bigcap_{k=0}^{\infty}]k, +\infty[= \emptyset$*, and hence*

$$\mu(\bigcap_{k=0}^{\infty}]k, +\infty[) = \mu(\emptyset) = 0 \neq lim_{k \to \infty} \mu(]k, +\infty[) = +\infty.$$

1.7 Measurable sets and topology. $\sigma-$algebras

Theorem 8. *(Measurable sets and topology)*

Every open subset and, hence, every closed subset in $\mathbb{R}^n$ *is measurable.*

Example 6. *Consider the set* $A = \{(x, y) \in \mathbb{R}^2; y \in \mathbb{Q}\}$.

Since $A = \bigcup_{q \in \mathbb{Q}} \{(x, y) \in \mathbb{R}^2; y = q\}$ *(countable union!) and each member of the union is measurable (since it is closed) with zero measure, it follows that the set* A *is measurable with zero measure.*

A family $\mathcal{E} \subseteq \mathbf{P}(\mathbb{R}^n)$ of subsets of $\mathbb{R}^n$ is said to be a $\sigma-$*algebra* if and only if the following conditions are satisfied:

- $\emptyset \in \mathcal{E}$.
- if $A \in \mathcal{E}$, then $A^c_{\mathbb{R}^n} \in \mathcal{E}$.
- Let $\{A_k; k \in \mathcal{A}\}$ be an at most countable family of subsets $A_k \in \mathcal{E}$. Then $\bigcup_{k \in \mathcal{A}} A_k \in \mathcal{E}$

IN PLAIN WORDS, a $\sigma-$algebra is family of subsets that contains the empty set $\emptyset$, and that is *"stable "* with respect to the set-theoretic operations of passage to the complemntary set, and at most countable union and intersection (by De Morgan laws).

Hence, the properties of the family of measurable sets can be summarized as follows:

The family of measurable subsets of $\mathbb{R}^n$ is a $\sigma-$algebra.

1.8 $\sigma-$algebras generated by a family of subsets. The Borel $\sigma-$algebra of $\mathbb{R}^n$

Let $\{\mathcal{E}_i; i \in \mathcal{A}\}$ *any* family of $\sigma-$algebras of $\mathbb{R}^n$.

Then, the intersection

$$\bigcap_{i\in\mathcal{A}} \mathcal{E}_i$$

is a $\sigma-$algebra of $\mathbb{R}^n$.

(The proof is almost immediate by the definition).

Definition 3. *(generated $\sigma-$algebre)*

Denote by $\mathcal{I}$ a family of subsets of $\mathbb{R}^n$.

By definition, we set:

$$\mathcal{S}_{\mathcal{I}} = \bigcap_{\mathcal{E}\supseteq\mathcal{I}} \mathcal{E}, \quad \mathcal{E} \quad \sigma - algebra \quad of \quad \mathbb{R}^n.$$

Since any intersection of $\sigma-$algebras is a $\sigma-$algebra,

$$\mathcal{S}_{\mathcal{I}} \quad is \quad a \quad \sigma - algebra,$$

called the *$\sigma-$algebra generated by the family $\mathcal{I}$.*

Clearly, $\mathcal{S}_{\mathcal{I}}$ is the smallest (in the sense of inclusion) $\sigma-$algebra that contains the family $\mathcal{I}$.

Remark 5. *(Identity principle for generated $\sigma-$algebras)*

Lat $\mathcal{I},\ \mathcal{J} \subseteq \mathbf{P}(\mathbb{R}^n),\ \ \mathcal{S}_{\mathcal{I}},\ \mathcal{S}_{\mathcal{J}}$ denote the $\sigma-$algebras generated by $\mathcal{I},\ \mathcal{J}$, respectively.

Then $\mathcal{S}_{\mathcal{I}} = \mathcal{S}_{\mathcal{J}}$ if and only if

$$\mathcal{I} \subseteq \mathcal{S}_{\mathcal{J}} \quad e \quad \mathcal{J} \subseteq \mathcal{S}_{\mathcal{I}}.$$

Definition 4. *(the Borel $\sigma-$algebra)*

Denote by $\mathcal{O}$ the family of all open subsets of $\mathbb{R}^n$.

By definition, we set:

$$\mathcal{B} =^{def} \mathcal{S}_{\mathcal{O}} = \bigcap_{\mathcal{E} \supseteq \mathcal{O}} \mathcal{E}, \quad \mathcal{E} \quad \sigma - algebra \quad of \quad \mathbb{R}^n.$$

The $\sigma-$algebra $\mathcal{B}$ generated by the family $\mathcal{O}$ of all open subsets of $\mathbb{R}^n$ is called the BOREL $\sigma-$algebra . A subset $A \subseteq \mathbb{R}^n$ is said to be a BORELIAN if and only if it belongs to $\mathcal{B}$.

From the "identity principle", the Borel $\sigma-$algebra $\mathcal{B}$ is also the $\sigma-$algebra generated by the family $\mathcal{C}$ of all closed subsets of $\mathbb{R}^n$.

Remark 6.

- *Any Borelian is a measurable subset of $\mathbb{R}^n$.*
- *(NON TRIVIAL) In general, it is false that a measurable subset of $\mathbb{R}^n$ is a Borelian.*
- *By definition, $\mathcal{B}$ is the smallest (in the sense of inclusion) $\sigma-$algebra that contains all the ope and the closed subsets of $\mathbb{R}^n$.*

IN PRACTICE, in order to prove that a subset $A \subseteq \mathbb{R}^n$ is Borelian (and,a fortiori, measurable), it is sufficient to prove that A can be expressed, starting with open and closed subsets (that Borelian by definition), just by using tht set-theoretic operations of passage to the complementary subset, set-theoretic difference and at most countable union and intersection.

Example 7. *Consider the set*

$$A = \{(x, y) \in \mathbb{R}^2; (x, y) \notin \mathbb{Q}^2\},$$

that is, the set of points of the plane $\mathbb{R}^2$ that have both the coordinates IRRATIONAL.

A is measurable A is Borelian? If YES, compute the measure of A.

Let us proceed as follows.

First, we notice that A can be expressed in the following way:

$$A = \mathbb{R}^2 - \{(x, y) \in \mathbb{R}^2; x, y \in \mathbb{Q}\} = \mathbb{R}^2 - \bigcup_{(x,y) \in \mathbb{Q} \times \mathbb{Q}} \{(x, y)\}.$$

now, $\mathbb{Q}$ is countable and, therefore, $\mathbb{Q} \times \mathbb{Q}$ is countable (Cantor theorem).

Hence

$$\bigcup_{(x,y)\in\mathbb{Q}\times\mathbb{Q}} \{(x,y)\}$$

is a countable union of singletons, that are closed (and, then, Borelians), and has zero measure.

It follows that

$$\mathbb{R}^2 - \{(x,y) \in \mathbb{R}^2; x, y \in \mathbb{Q}\}$$

is a difference of Borelians and, hence, is a Borelian.

Furthermore, we claim that $\bigcup_{(x,y)\in\mathbb{Q}\times\mathbb{Q}} \{(x,y)\}$*, being a countable union of subsets of zero measure, has - in turn - zero measure.*

By the additivity property of disjoint measurable subsets, it follows that:

$$\mu(\mathbb{R}^2) = \infty = \mu(A) + \mu(\bigcup_{(x,y)\in\mathbb{Q}\times\mathbb{Q}} \{(x,y)\}) = \mu(A) + 0,$$

and, hence, $\mu(A) = \infty$*.*

1.9 La $\sigma-$algebra $\mathcal{B}(\mathbb{R})$ dei Boreliani di $\mathbb{R}$

The Borel $\sigma-$algebra of $\mathbb{R}$ deserves special attention, as we shall see.

First of all, we recall a beautiful result - that is also of independent interest - on open sets of $\mathbb{R}$.

Lemma 1. (*Lindelöf*)

Let $\mathcal{A}$ *be any family of open sets of i* $\mathbb{R}$*. There exist a COUNTABLE subfamily* $\{A_n \in \mathcal{A}; n \in \mathbb{N}\} \subseteq \mathcal{A}$ *such that*

$$\bigcup_{A\in\mathcal{A}} A = \bigcup_{n=0}^{\infty} A_n.$$

Proof. Let $U = \bigcup_{A\in\mathcal{A}} A$, and $x \in U$. There exists an open set $A \in \mathcal{A}$ such that $x \in A$ and, hence, an open interval I_x such that $x \in I_x \subseteq A$.

Recall that: *given two real numbers* $\alpha < \beta$*, there exists a rational* q *such that* $\alpha < q < \beta$*.*

Hence, we can construct an open interval J_x with rational extrema such that $x \in J_x \subseteq I_x$. Since the set of all open interval with rational extrema is COUNTABLE, the family $\{J_x; x \in U\}$ is countable and $U = \bigcup_{x\in U} J_x$.

For every interval J_x, choose an $A \in \mathcal{A}$ that contains it. By this process, we obtain a countable subfamily $\{A_n \in \mathcal{A}; n \in \mathbb{N} \subseteq \mathcal{A}\}$ such that $U = \bigcup_{n=0}^{\infty} A_n$. $\square$

Proposition 3. *The following $\sigma-$algebras of $\mathbb{R}$ are equal:*

1. *The $\sigma-$algebra generated by the family* $\{[a, +\infty[; a \in \mathbb{R}\}$.
2. *The $\sigma-$algebra generated by the family* $\{]a, +\infty[; a \in \mathbb{R}\}$.
3. *The $\sigma-$algebra generated by the family* $\{[a, b]; a, b \in \mathbb{R}\}$.
4. *The $\sigma-$algebra generated by the family* $\{]-\infty, b]; b \in \mathbb{R}\}$.
5. *The $\sigma-$algebra generated by the family* $\{]-\infty, b[; b \in \mathbb{R}\}$.
6. *The $\sigma-$algebra generated by the family* $\{]a, b[; a, b \in \mathbb{R}\}$.
7. *The Borel $\sigma-$algebra $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$.*

Proof. The equality among the generated $\sigma-$algebras from 1. to 6. immediately from the "'identity principle": indeed, we one has to recognize from 1. to 6. that any element of a "'family of generators" from 1. to 6. can be obtained from the elements of other "'families of generators" by "'$\sigma-$algebra operations", that is, "at most countable unions/intersections", "passagge to the complementary set", "set difference" (simple exercise). Let denote by the simple $\mathcal{S}$ the $\sigma-$algebra

generated by the families da 1. – 6.. Since these familes are subfamilies either of the family $\mathcal{O}$ of all open sets or of the family $\mathcal{C}$ of all closed sets of $\mathbb{R}$, it follows that $\mathcal{S} \subseteq \mathcal{B}(\mathbb{R})$.

Conversely, by *Lindelöf* Lemma, any open subset of $\mathbb{R}$ is representable as COUNTABLE union of open intervals (generators 6,), hence, $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{S}$.

Therefore $\mathcal{B}(\mathbb{R}) = \mathcal{S}$. □

1.10 Borelians, measurable sets and inner measure

As we have seen, the outer measure of a subset of *mathbbR*^{n} is defined by a sort of "approximation from outside" procedure. (strictly speaking, considering the lower extreme in *overline mathbbR*) with "elementary" subsets, that is to say, unions of families with the most numberable of intervals containing the given set (lebesguian coverings).

It is natural to try to re-propose the same procedure "from the inside" - by defining a *itmeasure internal*, so that, for measurable subsets, external measure and internal coincide (in analogy with the notion of Riemmann integral). If we limit ourselves to still consider families of intervals, the one just outlined and 'the idea behind the theory of Peano / Jordan (which historically precedes the theory of Lebesgue / Borel); unfortunately, following this kind of approach, the class of measurable sets would be very narrow: as a result, we will define the *itmeasure internal* by an "approximation from the inside" through measurable or Borelian.

The inspiration for this strategy is suggested by the following fundamental result, whose proof is omitted.

Theorem 9. *Let $A \subseteq \mathbb{R}^n$.*

- $\mu^*(A) = inf\{\mu(E); E \text{ } measurable, E \supseteq A\} = inf\{\mu(O); O \text{ } open, O \supseteq A\}$.
- *There exists a borelian B such that $B \supseteq A$, $\mu^*(A) = \mu(B)$ e $\mu(C) = 0$ for every C measurable, $C \subseteq B - A$.*

Motivated by the preceding result, we define the *INNER MEASURE* of A by setting:

$$\mu_*(A) = sup\{\mu(E); E \text{ } measurable, E \subseteq A\}.$$

Theorem 10. *Let $A \subseteq \mathbb{R}^n$.*

There exists a borelian B such that $B \subseteq A$, $\mu^*(A) = \mu(B)$ and $\mu(C) = 0$ for every C measurable, $C \subseteq A - B$.

Theorem 11. *A subset $A \subseteq \mathbb{R}^n$ is mesurable if and only if, for every $\varepsilon \in \mathbb{R}^+$ there exist an open set O and a closed set K such that $K \subseteq A \subseteq 0$ e $\mu(O - K) < \varepsilon$.*

Remark 7.

- *For every $A \subseteq \mathbb{R}^n$, we have $\mu_*(A) \leq \mu^*(A)$.*

  Indeed, if E, F are measurable and $E \subseteq A \subseteq F$, then $\mu(E) \leq \mu(F)$ and, hence,

  $$\mu_*(A) = sup\{\mu(E); E \text{ } measurable, E \subseteq A\} \leq \mu(F)$$

  for every F measurable, $F \supseteq A$. Therefore,

  $$\mu_+(A) \leq inf\{\mu(F); A \subseteq F\} = \mu^*(A).$$

- *If A is measurable, then $\mu_*(A) = \mu(A) = \mu^*(A)$.*

  Indeed, let B be a Borelian such that $B \subseteq A$ and $\mu(B) 0 \mu_*(A)$.

  Since A is measurable, then $A - B$ is measurable and furthermore $\mu(A - B) = 0$.

  Therefore

  $$\mu^*(A) = \mu(A) = \mu(B) + \mu(A - B) = \mu(B) = \mu_*(A).$$

- *"Conversely", let* $A \subseteq \mathbb{R}^n$ *and let* $\mu_(A) = \mu^*(A) < +\infty$*. Then* A *is measurable.*

  Indeed, we know that here exist two borelians E, F *such that* $E \sqsubseteq A \subseteq F$ *and*

  $$\mu(E) = \mu_*(A), \quad \mu^*(A) = \mu(F).$$

  From $F = E \cup (F - E)$*, it follows that* $\mu(F) = \mu(E) + \mu(F - E)$ *and, since* $\mu_(E) = \mu(F) < +\infty$*, we infer* $\mu(F - E) = 0$*.*

  On the other hand, given an arbitrary $\varepsilon \in \mathbb{R}^+$*, there exists a closed set* K *such that* $K \subseteq E$ *and* $\mu(E - K) < \frac{\varepsilon}{2}$*. (Ideed* E *borelian implies* E *measurable. Then, there exists an open subset* E'*,* $E' \supseteq E$ *such that* $\mu(E' - K) < \frac{\varepsilon}{2}$*; since* $E - K \subseteq E' - K$*, then* $\mu(E - K) < \frac{\varepsilon}{2}$*.)*

  From similar argument, there exists an open set O*,* $O \supseteq F$ *such that* $\mu(O - F) < \frac{\varepsilon}{2}$*.*

  Since $O - F = (O - F) \cup (F - E) \cup (E - K)$*, risulta* $\mu(O - K) < \varepsilon$*, and the assertion follows from the preceding result.*

1.11 Transformation of coordinates and invariance properties of the measure

Theorem 12. *Let L be an invertible linear operator from $\mathbb{R}^n$ to itself,* $\mathbf{c}$ *a vector in* $\mathbb{R}^n$.

Consider the (affine) transformation $T(x) = L(x) + \mathbf{c}$, for every $x \in \mathbb{R}^n$.

Foe every $A \subseteq \mathbb{R}^n$, the set $T[A] = \{T(x);\ x \in A\}$ is measurable if and only if A is measurable.

Furthermore $\mu(T[A]) = |det(\mathcal{M})| \cdot \mu(\mathcal{A})$, where $\mathcal{M}$ denotes the matrix of L with respect to a given pair of bases.

In particular, if$T$ is an ISOMETRY (cioe' $det(\mathcal{M}) = \pm 1$), then $\mu(T[A]) = \mu(A)$.

In particular, it follows that the measue is invariant with respect to euclidean movements, that is $det(\mathcal{M}) = 1$ (rototranslations).

2 Measurable function and ed Lebesgue integrals

2.1 Measurable function

It is useful to extend the system of real numbers "adding" two formal elements $+infty, -infty$. We will again indicate this system by the symbol $\overline{\mathbb{R}}$, and call it the system of *exteded real extended*.

Consistently, the order definition is extended by placing $-\infty < x < +infty$, for every $x\ in\ mathbbR$.

We also set $x+\infty = +\infty$, $x-\infty = -\infty$, $x\ cdot\infty = +\infty$, $x\ cdot(-\infty) = -\infty$ if $x > 0$, $\infty + \infty = +\infty$, $-\infty - \infty = -\infty$, $\infty\ cdot(\ pm\infty) = \ pm\infty$.

The "operation" $\infty - \infty$ is left INDEFINITE, while we will adopt the convention $0\ cdot\infty = 0$.

One of the uses of the extended real numbers is in the use of the expression $sup\ S$. If S is a set $\ underlinenot\ empty$ of real numbers on the upper limit, we remind you that $sup\ S$ exists and is the minimum of the major of S. If S is not superiorly bounded, then $sup\ S = +\ infty$. If we put $sup\ \ emptyset = -\ infty$, then, in all cases $sup\ S$ remains defined as the smallest extended real number which is greater than or equal to each element of the set S.

A function that takes values in $\overline{\mathbb{R}}$ is called function with <u>*extended real values*</u>.

Proposition 4. *Let $f : A \to \overline{\mathbb{R}}$ be function with <u>extended real values</u>, $A \subseteq \mathbb{R}^n$, A measurable. The following assertions are equivalent:*

1. *for every real number α, the set $\{x \in A; f(x) > \alpha\}$ is measurable;*
2. *for every real number α, the set $\{x \in A; f(x) \geq \alpha\}$ is measurable;*
3. *for every real number α, the set $\{x \in A; f(x) < \alpha\}$ is measurable;*
4. *for every real number α, the set $\{x \in A; f(x) \leq \alpha\}$ is measurable.*

Proof.

- $(1 \Rightarrow 4)$

  Indeed, $\{x \in A; f(x) \leq \alpha\} = A - \{x \in A; f(x) > \alpha\}$ and we know that the set-difference of measurable sets is measurable.

- For the same reasons $(4 \Rightarrow 1)$ and, furthermore, $2 \Leftrightarrow 3$.

- $(1 \Rightarrow 2)$

  Since $\{x \in A; f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x \in A; f(x) > \alpha - \frac{1}{n}\}$, the set $\{x \in A; f(x) \geq \alpha\}$ is a countable intersection of measurable sets.

- $(2 \Rightarrow 1)$

  since $\{x \in A; f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x \in A; f(x) > \alpha + \frac{1}{n}\}$, the set $\{x \in A; f(x) \geq \alpha\}$ is a countable union of measurable sets.

$\square$

FUNDAMENTAL DEFINITION

A function $f : A \to \overline{\mathbb{R}}$ with <u>*extended real values*</u>, $A \subseteq \mathbb{R}^n$, is said to be <u>*MEASURABLE*</u> if and only if:

- A is measurable;
- f satisfy (one of the) four equivalent conditions $1), 2), 3), 4)$.

From Proposition 3 (sect. 1.9) we infer:

Proposition 5. *Let $f : A \to \overline{\mathbb{R}}$ be a function with extended real values, $A \subseteq \mathbb{R}^n$, A measurable. The following assertions are equivalent:*

- *f is measurable;*
- *for every $A \subseteq \mathbb{R}$ open, the set $f^{-1}[A]$ is measurable;*
- *for every $C \subseteq \mathbb{R}$ closed, l' insieme $f^{-1}[C]$ is measurable.*

Corollary 3. *Let $f : A \to \mathbb{R}$ be a continuous function, A measurable. The f is measurable.*

Proof. Let $\alpha \in \mathbb{R}$. By definition, we have, $\{x \in A; f(x) > \alpha\} = f^{-1}(\ ]\alpha, +\infty[\ )$. Since $]\alpha, +\infty[$ is open and f is continuous, there exist an open set $B \subseteq \mathbf{R}^n$ such that $\{x \in A; f(x) > \alpha\} = f^{-1}(\ ]\alpha, +\infty[\ ) = B \cap A$. Since A is measurable, and B is measurable since it is open, then $\{x \in A; f(x) > \alpha\}$ is measurable. $\square$

Remark 8. *L The class of measurable function on a measurable set S is "much larger" than the set of continuous functions on A. For example, consider the " Dirichlet function" $\chi : [0, 1] \to \mathbb{R}$ defined as follows:*

$$\chi(x) = 0 \ \forall x \in [0, 1] \cap \mathbb{Q}, \qquad \chi(x) = 1 \ \textit{ otherwise.}$$

The function χ is clearly EVERYWHERE DISCONTINUOUS on its closed domain $[0, 1]$; however, since $[0, 1] \cap \mathbb{Q}$ and $[0, 1] \cap (\mathbb{R} - \mathbb{Q})$ are measurable, the function χ is measurable.

Remark 9. *The is "consistency" between the notion of measurable set and the notion of measurable function.*

Indeed, give a subset $A \subseteq \mathbb{R}^n$ and denoted by $\chi_A : \mathbb{R}^n \to \{0,1\}$ its characteristic function, it is a simple exercise to prove that A is measurable (as a set) if and only if χ_A is measurable (as a function).

The class of measurable functions is "stable" with respect to algebraic operations.

Proposition 6. *let $f, g : A \to \mathbb{R}$ be real valued functions, A measurable. let $c \in \mathbb{R}$. Then $f+g, c \cdot f, f+c, f-g, f \cdot g$ are measurable.*

A crucial difference between the class of measurable functions and the class of continuous functions is that the first - unlike the second - is "stable" with respect to "order" operations, such as "*sup*", *inf*", "*minlim*", "*maxlim*" and pointwise limit $lim_{n\to\infty}$.

Definition 5. *Let $\{f_1, \ldots, f_n\}$, $(f_n)_{n\in\mathbb{N}}$ be a finite family and a sequence of measurable functions with the same domain $A \subseteq \mathbb{R}^n$, respectively. We set:*

- $$inf\{f_1, \ldots, f_n\} : A \to \overline{\mathbb{R}},\ inf\{f_1, \ldots, f_n\}(x) = inf\{f_1(x), \ldots, f_n(x)\}\ \forall x \in A;$$

- $$sup\{f_1, \ldots, f_n\} : A \to \overline{\mathbb{R}},\ sup\{f_1, \ldots, f_n\}(x) = sup\{f_1(x), \ldots, f_n(x)\}\ \forall x \in A;$$

- $$inf(f_n)_{n\in\mathbb{N}} : A \to \overline{\mathbb{R}},\ inf(f_n)_{n\in\mathbb{N}}(x) = inf(f_n(x))_{n\in\mathbb{N}}\ \forall x \in A;$$

- $$sup(f_n)_{n\in\mathbb{N}} : A \to \overline{\mathbb{R}},\ sup(f_n)_{n\in\mathbb{N}}(x) = sup(f_n(x))_{n\in\mathbb{N}}\ \forall x \in A;$$

- $$maxlim(f_n)_{n\in\mathbb{N}} =^{def}\ \ inf_{n\in\mathbb{N}}\ (sup(f_k)_{k\geq n}\ )$$

  $$minlim(f_n)_{n\in\mathbb{N}} =^{def}\ \ sup_{n\in\mathbb{N}}\ (inf(f_k)_{k\geq n}\ );$$

- *By definition, it follows that*

  $$minlim(f_n)_{n\in\mathbb{N}} \leq maxlim(f_n)_{n\in\mathbb{N}};$$

if (and only if)

$$minlim(f_n)_{n\in\mathbb{N}} = maxlim(f_n)_{n\in\mathbb{N}}$$

the sequence $(f_n)_{n\in\mathbb{N}}$ *is pointwise convergent, and the common value*

$$minlim(f_n)_{n\in\mathbb{N}} = maxlim(f_n)_{n\in\mathbb{N}} =^{def} lim_{n\to\infty}\ f_n$$

is the point limit of the sequence of functions $(f_n)_{n\in\mathbb{N}}$.

Theorem 13. *Let* $\{f_1,\dots,f_n\}$, $(f_n)_{n\in\mathbb{N}}$ *be a finite family and a sequence of measurable functions with the same domain* $A\subseteq\mathbb{R}^n$, *respectively.*

Then, the functions $inf\{f_1,\dots,f_n\}$, $sup\{f_1,\dots,f_n\}$, $inf(f_n)_{n\in\mathbb{N}}$, $sup(f_n)_{n\in\mathbb{N}}$, $minlim(f_n)_{n\in\mathbb{N}}$, $maxlim(f_n)_{n\in\mathbb{N}}$ *are measurable.*

In particular, if the sequence of functions $(f_n)_{n\in\mathbb{N}}$ *is pointwise convergent, then* $lim_{n\to\infty}\ f_n$ *is a measurable function.*

Proof. Notice that, for every $\alpha\in\mathbb{R}$;

$$\{x\in A; sup\{f_1,\dots,f_n\}(x)>\alpha\} = \bigcup_{i=1}^{n}\{x\in A; f_i(x)>\alpha\};$$

and, hence, the measurability of the functions $f_1,\dots,f_n$ implies the measurability of the function $sup\{f_1,\dots,f_n\}$.

Similarly,

$$\{x\in A; sup(f_n)_{n\in\mathbb{N}}(x)>\alpha\} = \bigcup_{n=1}^{\infty}\{x\in A; f_n(x)>\alpha\},$$

and, hence, the function $sup(f_n)_{n\in\mathbb{N}}$ is measurable.

In a parallel way, we proceed with the function "inf".

The remaining assertions directly follow from the definitions. $\square$

Remark 10. *The preceding result is FALSE if we replace "measurable function" by "continuous functions".*

For example, consider the sequence of functions

$$f_n:[0,1]\to\mathbb{R},\qquad f_n(x)=x^n\quad\forall x\in[0,1].$$

The functions f_n *are polynomial and , hence, continuous (e limitate). The sequence* $(f_n)_{n\in\mathbb{N}}$ *is pointwise convergent*

$$lim_{n\to\infty}\ f_n = f,$$

being the pointwise limit the function $f : [0,1] \to \mathbb{R}$ *such that*

$$f(x) = 0 \ \ \forall x \in [0,1[, \qquad f(1) = 0.$$

Then, the pointwise limit f *is discontinuous at the point* $x = 1$.

However f *is still measurable! (see, e.g., the next result).*

Definition 6. *A property (P) is said to be true ALMOST EVERYWHERE (abbr. A.E.) if and only if the set of point in which it is FALSE has zeromeasure.*

For example, given two functions f, g *with the same measurable domain* A*, we set* $f = gA.E.$ *if and only if* $\mu(\{x \in A; f(x) \neq g(x)\}) = 0$.

Similarly, we say that a sequence of functions $(f_n)_{n\in\mathbb{N}}$ *with domain* A *id pointwise convergent A.E.to a function* f *if and only if there exists a subset* $E \subseteq A$*,* $\mu(E) = 0$*, such that* $(f_n)_{n\in\mathbb{N}}$ *is pointwise convergent to* f *on the domian* $A - E$*, that is*

$$lim_{n\to\infty} f_n(x) = f(x), \qquad x \in \forall A - E.$$

Proposition 7. *If* f *is a measurable function on the domain* A *and* $g = fA.E.$ *on* A*, then* g *is measurable.*

Proof. Let $E = \{x \in A; f(x) \neq g(x)\}$, $\mu(E) = 0$. Now

$$\{x \in A; g(x) > \alpha\} = (\{x \in A; f(x) > \alpha\} \cup \{x \in E; g(x) > \alpha\}) - \{x \in E; g(x) \leq \alpha\}.$$

The set $\{x \in A; f(x) > \alpha\}$ is measurable, since f is measurable. The sets $\{x \in E; g(x) > \alpha\}$, $\{x \in E; g(x) \leq \alpha\}$ are measurable, since subsets of the set E that has zero measure . □

Remark 11. *The Dirichlet function (Remark* **??***) is measurable, since it is equal A.E. to the constant function* 1.

The pointwise limit f *of Remark* **??** *is measurable, since it is equal A.E. to the constant function* 0.

The next results may be "informally / intuitively" summarized by stating (Littlewood) that every pointwise convergent sequence of measurable functions (on domains of *underlinefinite measure*) is "almost" uniformly convergent.

Proposition 8. *(Littlewood Lemma)*

Let $A \subseteq \mathbb{R}^n$*,* $\mu(A) < \infty$*. Sia* $(f_n)_{n\in\mathbb{N}}$ *be a sequence of measurable functions with domain* A

and let $f : A \to \mathbb{R}$ be a real valued function such that

$$lim_{n\to\infty} f_n(x) = f(x), \qquad \forall x \in A.$$

Then, for every $\varepsilon \in \mathbb{R}^+$ and for every $\delta \in \mathbb{R}^+$, there exists a measurable subset $B \subseteq A$, $\mu(B) < \delta$ and a posite integer N such that

$$|f_n(x) - f(x)| < \varepsilon, \qquad \forall x \notin B, \qquad \forall n > N.$$

A more intuitive reformulation of the previous result is given by the following statement.

Theorem 14. *(Egoroff) Let $A \subseteq \mathbb{R}^n$, $\mu(A) < \infty$. Let $(f_n)_{n\in\mathbb{N}}$ a sequence of measurable functions with domain A and let $f : A \to \mathbb{R}$ a function to real values such that*

$$lim_{n\to\infty} f_n(x) = f(x), \qquad \forall x \in A \qquad (\textit{pointwise convergence in } A).$$

Then, for every $\eta \in \mathbb{R}^+$ (arbitrarily small) there is a measurable subset

$$B \subseteq A, \ \mu(B) < \eta$$

such that $(f_n)_{n\in\mathbb{N}}$ is uniform convergent to f over $A - B$.

Remark 12. *Recall that, given a sequence of real valued functions $(f_n)_{n\in\mathbb{N}}$ with domain $X \subseteq \mathbb{R}^n$, it is said that it is $\underline{uniform\ convergent}$ to the function $f : X \to \mathbb{R}$ if*

$$\forall \varepsilon \in \mathbb{R}^+, \quad \exists N_\varepsilon \in \mathbb{N}$$

such that

$$|f_n(x) - f(x)| < \varepsilon, \qquad \forall n > N_\varepsilon \qquad \forall x \in X.$$

2.2 The Riemann integral

Given a closed interval $[a, b] \subseteq \mathbb{R}$, a *STEP FUNCTION* on $[a, b]$ is a function $\Psi : [a, b] \to \mathbb{R}$ of the following form:

$$\Psi(x) = c_i, \ \ \forall x \in]x_{i-1}, x_i], \qquad c_i \in \mathbb{R}, \qquad\qquad (\ddagger)$$

for some *subdivision* $a = x_0 < x_1 < x_2 < \cdots x_n = b$ of the closed interval $[a, b]$, and a set of constants $\{c_1, c_2, \dots, c_n\}$. Equivalently,

$$\Psi = \Psi(a) \cdot \chi_{\{a\}} + \sum_{i=1}^{n} c_i \cdot \chi_{]x_{i-1}, x_i]},$$

where χ are characteristic functions of intervals.

l

Given a step function $\psi : [a,b] \subseteq \mathbb{R}$ of the form (‡), the integral of Ψ of $[a,b]$ is (naturally) definited as follows:

$$\int_a^b \Psi(x)dx = \sum_{i=1}^{n} c_i \cdot (x_i - x_{i-1}).$$

Given a limited function $f : [a,b] \to \mathbb{R}$, the *Riemann lower integral* of f on $[a,b]$ is defined as follows::

$$\underline{R}\int_a^b f(x)dx = sup\ _{\Psi \le f}\ \int_a^b \Psi(x)dx,$$

where *sup* is taken with respect to all step functions $\Psi \le f$ defined on the interval $[a,b]$.

Given a limited function $f : [a,b] \to \mathbb{R}$, the *Riemann upper integral* di f su $[a,b]$ is defined as follows::

$$\underline{R}\int_a^b f(x)dx = inf\ _{\Phi \le f}\ \int_a^b \Psi(x)dx,$$

where *sup* is taken with respect to all step functions $\Psi \ge \Phi f$ defined on the interval $[a,b]$.

By the definitions, we have:

$$\underline{R}\int_a^b f(x)dx \le \overline{R}\int_a^b f(x)dx.$$

Given a limited function $f : [a,b] \to \mathbb{R}$, we will say that it is *Riemann integrable* (abbr., $R-$ *integrable*) on $[a,b]$ if and only if their lower and upper integrals are *equal*, and its Riemann integral $R\int_a^b f(x)dx$ di f is, by definition, the common value. In symbols,

$$R\int_a^b f(x)dx =^{def}\ \underline{R}\int_a^b f(x)dx = \overline{R}\int_a^b f(x)dx.$$

We denote by the symbol $R\int_a^b f(x)dx$ the Riemann integral to distinguish it from the Lebesgue integral, that will be discussed in the following.

Theorem 15. *(Lebesgue-Vitali)*

A limited function $f : [a,b] \to \mathbb{R}$, $f : [a,b] \to \mathbb{R}$ is $R-$integrable on the interval $[a,b]$ if and only if f is continuous A.E. on $[a,b]$.

2.3 The Lebesgue integral of simple functions

Let $E \subseteq \mathbb{R}^n$ and denote by χ_E its characteristic function.

A linear combination

$$\varphi = \sum_{i=1}^{m} c_i \cdot \chi_{E_i}, \qquad c_i \in \mathbb{R} \qquad\qquad (\dagger)$$

is said to be a SIMPLE FUNCTION if and only if $E_i,\ i = 1, \ldots, m$ are measurable and $\mu(E_i) < \infty,\ i = 1, \ldots, m$

Remark 13. • *Any step function is a measurable function.*

- *The representation of the form* $(\dagger)$ *IS NOT UNIQUE.*
- *a function* φ *is simple if and only if it is measurable, its "support" supp* $\varphi = \{x \in \mathbb{R}^n; \varphi(x) \neq 0\}$ *has finite measure, and* φ *assumes only a finite set* $\{a_1, \ldots, a_n\}$ *of non zero values.*

Let φ be a simple function, sand let $\{a_1, \ldots, a_n\}$ be the finite set of non zero values that it assumes; then

$$\varphi = \sum_{i=1}^{n} a_i \cdot \chi_{A_i} \qquad\qquad (\dagger\dagger),$$

where

$$A_i = \{x \in \mathbb{R}^n; \varphi(x) = a_i\}.$$

The representation $(\dagger\dagger)$ of φ is called its *canonical representation*, is clearly unique and is, in turn, characterized by the following conditions:

- the (non empty, measurable) sets A_i are pairwise disjoint;
- the coefficients $\{a_1, \ldots, a_n\}$ are pairwise different and different from zero.

THE INTEGRAL of the simple function φ is defined as follows:

Definition 7. • *Let*

$$\int \varphi = \sum_{i=1}^{n} a_i \cdot \mu(A_i), \qquad\qquad (\S),$$

where $\sum_{i=1}^{n} a_i \cdot \chi_{A_i}$ *is the canonical representation of* φ.

- *Let $E \subseteq \mathbb{R}^n$ be measurable, $\mu(E) < \infty$. The integral on E of the simple function φ is:*
$$\int_E \varphi = \int \varphi \cdot \chi_E = \sum_{i=1}^{n} a_i \cdot \mu(A_i \cap E). \qquad (\S\S)$$

At this point, the integral of a simple function is correctly (univocally) defined in terms of the canonical representation of the function.

An important question that arises naturally is the following; "if the representation we know is not the canonical one, how to calculate the integral"?

The next result, the proof of which is a matter of simple exercise, gives us - among other consequences - the answer.

Proposition 9. *Letφ, ψ be simple functions on $\mathbb{R}^n$. Let $a, b \in \mathbb{R}$. Then*
$$\int (a\phi + b\psi) = \int a\phi + \int b\psi \qquad (linearity).$$
Furthermore, if $\varphi \geq \psi$ A.E., then
$$\int \varphi \geq \int \psi \qquad (monotonicity).$$

Remark 14. *The first statement of the previous Proposition expresses the linearity of the operation of the transition to the integral. This implies, as a relevant consequence, the possibility of calculating the integral of a simple function regardless of the knowledge of the canonical representation.*

Notice that $\int \chi_E = \mu(E)$.)

Let
$$\varphi = \sum_{i=1}^{m} c_i \cdot \chi_{E_i}, \qquad c_i \in \mathbb{R} \qquad (\dagger),$$
where($\dagger$) is not the canonical representation. By the linearity property, it is still true that
$$\int \varphi = \sum_{i=1}^{n} c_i \cdot \mu(E_i).$$

2.4 The Lebesgue integral for limitated functions on domains of finite measure

Let $f : E \to \mathbb{R}$ be a limited function $E \subseteq \mathbb{R}^n$, $\mu(E) < \infty$.

In analogy with what has been done for the Riemann integral, let's consider the real numbers:

- $$sup\ _{\psi \leq f} \int_E \psi,$$

  where *sup* is considered with respect to all ψ the *simple* such that $\psi(x) \leq f(x), \forall x \in E$.

- $$inf\ _{\varphi \geq f} \int_E \varphi,$$

  where *inf* is considered with respect to all ψ the *simple* such that $\varphi(x) \geq f(x), \forall x \in E$.

It is immediate that:

$$sup\ _{\psi \leq f} \int_E \psi \leq inf\ _{\varphi \geq f} \int_E \varphi.$$

Definition 8. *Given a limited function $f : E \to \mathbb{R}, \mu(E) < \infty$, we will say that it is Lebesgue integrable (abbr., $L-$ integrable) on E if and only if the equality holds, and the Lebesgue integral $\int_E f\ f$ is, by definition, the common value. In symbols*

$$\int_E f = sup\ _{\psi \leq f} \int_E \psi = inf\ _{\varphi \geq f} \int_E \varphi.$$

The following fundamental result characterizes very much clear and elegant features of $f : E \to \mathbb{R}$, ($\mu(E) <\ infty,\ f\ limited$) which are $L-$ integrable. Notice that, comparing with the Lebesgue-Vitali Theorem, how much the new class of integrable functions will be "wider" (also in the case $E = [a, b]\ subseteq \mathbb{R}$).

Theorem 16. *Let $f : E \to \mathbb{R},\ \mu(E) < \infty$, be limited on E.*

The f is Lebesgue integrable on E if and only if it is measurable.

Proof. Assume that f is measurable and fix a positive number $M \in \mathbb{R}$ such that $|f(x)| < M$, $\forall x \in E$.

Hence, for every $n \in \mathbb{Z}^+$, the sets

$$E_k = \{x \in E;\ \frac{kM}{n} > f(x) \geq \frac{(k-1)M}{n}\ \}, \qquad k \in \mathbb{Z}, \qquad -n \leq k \leq n$$

are measurable, disjoint and their union equals E.

Recall that $\sum_{k=-n}^{n}\ \mu(E_k) = \mu(E)$.

The *simple* functions:

$$\varphi_n = \frac{M}{n} \cdot \sum_{k=-n}^{n} k \cdot \chi_{E_k},$$

$$\psi_n = \frac{M}{n} \cdot \sum_{k=-n}^{n} (k-1) \cdot \chi_{E_k}$$

satisfy the inequalities

$$\psi_n(x) \leq f(x) \leq \varphi_n(x), \quad \forall x \in E.$$

It follows that

$$sup_{\ \psi \leq f,\ \psi\ semplice} \int_E \psi \geq \int \psi_n = \frac{M}{n} \cdot \sum_{k=-n}^{n} (k-1) \cdot \mu(E_k),$$

$$inf_{\ \varphi \geq f,\ \varphi\ semplice} \int_E \varphi \leq \int \varphi_n = \frac{M}{n} \cdot \sum_{k=-n}^{n} k \cdot \mu(E_k),$$

and, hence,

$$0 \leq inf_{\ \varphi \geq f,\ \varphi\ semplice} \int_E \varphi - sup_{\ \psi \leq f,\ \psi\ semplice} \int_E \psi \leq \frac{M}{n} \cdot \sum_{k=-n}^{n} \mu(E_k) = \frac{M}{n} \cdot \mu(E).$$

Since $n \in \mathbb{Z}^+$ is arbitrary, we infer that

$$inf_{\ \varphi \geq f,\ \varphi\ semplice} \int_E \varphi - sup_{\ \psi \leq f,\ \psi\ semplice} \int_E \psi = 0,$$

and, then, we proved that the condition "f is measurable" is sufficient to the $L-$integrability. Viceversa, suppose that

$$inf_{\ \varphi \geq f,\ \varphi\ semplice} \int_E \varphi = sup_{\ \psi \leq f,\ \psi\ semplice} \int_E \psi.$$

Therefore, fixed a positive integer $n \in \mathbb{Z}^+$, there exist two simple functions ψ_n, φ_n such that

$$\psi_n(x) \leq f(x) \leq \varphi_n(x),\ \forall x \in E,$$

with

$$\int_E \varphi_n - \int_E \psi_n < \frac{1}{n}.$$

Recall that the functions

$$\psi^* = sup\ \psi_n \quad e \quad \varphi^* = inf\ \varphi_n$$

are measurable and

$$\psi^*(x) \leq f(x) \leq \varphi^*(x), \qquad \forall x \in E.$$

now, the set

$$\Delta = \{x \in E; \psi^*(x) \leq \varphi^*(x)\}$$

is the union of the subsets

$$\Delta_r = \{x \in E; \psi^*(x) < \varphi^*(x) - \frac{1}{r}\}, \qquad r \in \mathbb{Z}^+.$$

Every set Δ_r is contained in the set

$$\{x \in E; \psi_n(x) < \varphi_n(x) - \frac{1}{r}\}$$

and this has measure less than $\frac{r}{n}$. Since n is arbitrary, then $\mu(\Delta_r) = 0$, that implies $\mu(\Delta) = 0$.

Therefore, $\psi^* = \varphi^* A.E.$ and, a fortiori, $\psi^* = f = \varphi^* A.E.$; since ψ^*, φ^* are measurable, f is measurable, and the condition "f misurable" is necessary to the $L-$integrability on E.

□

Remark 15. *The previous result shows, inter alia, that the notion of Lebesgue integral for limited functions on domains of finite measurement it is consistent with that previously given for simple functions; in fact, we have observed that simple functions are measurable functions with finite measurement support and, clearly, the two notions of integral (when restricted to the case of simple functions) provide the same result.*

The following result shows that indeed the Lebesgue integral is a generalization - in the strict sense - of the Riemann integral.

Proposition 10. *Let f be a limited function defined on an interval $[a, b] \subseteq \mathbb{R}$. If f iso Riemann integrable on $[a, b]$, then f is Lebesgue integrable on$[a, b]$, and, hence, measurable. Furthermore*

$$R\int_a^b f(x)dx = \int_{[a,b]} f.$$

Proof. Notice that any step function h on $[a,b]$ uniquely defines a simple function ξ such that $\xi(x) = h(x), \forall x \in [a,b]$, $\xi(x) = 0, \forall x \notin [a,b]$, and viceversa. Then,we have

$$\underline{R}\int_a^b f(x)dx \leq sup_{\ \psi\leq f\psi,\psi\ simple} \int_{[a,b]} \psi \leq inf_{\ \varphi\geq f,\varphi\ simple} \int_{[a,b]} \varphi \leq \overline{R}\int_a^b f(x)dx.$$

Since f is Riemann integrable, the preceding inequalities are EQUALITIES, and, then, f is Lebesgue integrable and, furthermore.

$$R\int_a^b f(x)dx = \int_{[a,b]} f.$$

□

Proposition 11. *Let f, g be measurable functions, limited, defined on a domain E of finite measure. Then:*

- $$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g, \qquad \alpha, \beta \in \mathbb{R}.$$
- *Se $f = gA.E.$, then*
$$\int_E f = \int_E g.$$
- *Se $f \leq gA.E.$, then $\int_E f \leq \int_E g$. In particular, $|\int_E f| \leq \int_E |f|$.*
- *Let $k, K \in \mathbb{R}$ be such that $k \leq f(x) \leq K,\ \forall x \in E$. Then*
$$k \cdot \mu(E) \leq \int_E f \leq K \cdot \mu(E).$$
- *Let A, B be measurable of finite measure, $A \cap B = \emptyset$. Then*
$$\int_{A\cup B} f = \int_A f + \int_B f.$$

Example 8. *Consider the function $f : [0,1] \to \mathbb{R}$ defined as follows:*

$$f(x) = x^2,\ x \in [0,1] - \mathbb{Q}, \qquad f(x) = 0,\ x \in [0,1] \cap \mathbb{Q}.$$

The function f is everywhere discontinuous; however, it is measurable, since

$$f = gA.E., \qquad where \qquad g(x) = x^2.$$

It follows that f is $L-$integrable on $[0,1]$ and, furthermore:

$$\int_{[0,1]} f = \int_{[0,1]} g = R\int_0^1 x^2dx = [\frac{x^3}{3}]_{x=0}^{x=1} = \frac{1}{3}.$$

Proposition 12. *(dominated pointwise convergence)*

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions defined on a domain E of finite measure.

aussme that there exists a constant $M \in \mathbb{R}$ such that

$$|f(x)| \leq M, \qquad \forall x \in E, \qquad \forall n \in \mathbb{N}. \qquad (*)$$

If

$$lim_{n\to\infty} f_n(x) = f(x), \qquad \forall x \in E,$$

(that is, the sequence of functions in pointwise convergent to f, then

$$lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof. By Prop. 6 (Littlewood Lemma), for every $\varepsilon \in \mathbb{R}^+$, there exist a positive integer N and a measurable subset $B \subseteq E$, *con* $\mu(B) < \frac{\varepsilon}{4M}$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2 \cdot \mu(E)}, \qquad \forall x \in E - B, \qquad \forall n > N.$$

Hence

$$|\int_E f_n - \int_E f| = |\int_E (f_n - f)| \leq \int_E |f_n - f| \leq \int_{E-B} |f_n - f| + \int_B |f_n - f|.$$

Now

$$\int_{E-B} |f_n - f| < \frac{\varepsilon}{2 \cdot \mu(E)} \cdot \mu(E - B) < \frac{\varepsilon}{2},$$

$$\int_B |f_n - f| < 2M \cdot \frac{\varepsilon}{4M} = \frac{\varepsilon}{2},$$

that implies

$$|\int_E f_n - \int_E f| < \varepsilon \qquad \forall n > N.$$

Therefore

$$lim_{n\to\infty} \int_E f_n = \int_E f.$$

□

Remark 16. *If we neglect the hypothesis $(*)$ (called "dominance"), the above statement is FALSE.*

For example, we consider the sequence

$$(f_n)_{n\ in\mathbb{Z}^+}, \qquad f_n = n\cdot\chi_{[0,\frac{1}{n}]}.$$

The sequence does not satisfy the hypothesis $(*)$, *and converts to the identically zero function* $f\equiv 0$ *on the interval* $[0,1]$.

We have

$$\int_{[0,1]} f_n = n\cdot\mu([0,\frac{1}{n}]) = n\cdot\frac{1}{n} = 1, \qquad \int_{[0,1]} f = 0.$$

Then,

$$lim_{n\to\infty}\int_{[0,1]} f_n = 1 \neq 0 = \int_{[0,1]} f.$$

2.5 The Lebesgue integral for non-negative measurable functions

In this section, we plan to further extend the theory to the case of functions *not necessarily limited*, and on domains *not necessarily of finite measure*; however, we are forced to consider, for the time being, functions with *not negative* values.

Remark 17. *Let* $h: E\to\mathbb{R}$ *be a limited measurable function, and assume that its support* $supp(h) = \{x\in E; h(x)\neq 0\}$ *has finite measure.*

Then $\int_{supp(h)} h$ *is defined by the preceding sction, and we set, in a natural way,*

$$\int_E h =^{def} \int_{supp(h)} h.$$

Definition 9. *Let* $f: E\to\overline{\mathbb{R}}$ *be a non-negative measurable function. By definition, we set*

$$\int_E f =^{def} sup_{\ h\leq f}\int_E h,$$

where sup is considered with respect to all measurable, limited functions f, *with support of finite measure, and such that* $h(x)\leq f(x),\ \forall x\in E$.

Proposition 13. *Let* f, g *be non-negative measurable functions. Then:*

- $$\int_E c\cdot f = c\cdot\int_E f, \qquad \forall c\in\overline{\mathbb{R}}.$$

- $$\int_E (f+g) = \int_E f + \int_E g;$$

  furthermore, if the non- negative measurable g is such that $\int_E g < \infty$*, we have*

  $$\int_E (f-g) = \int_E f - \int_E g;$$

- *se $f \leq g$ A.E., then*

  $$\int_E f \leq \int_E g.$$

Theorem 17. *(Fatou Lemma)*

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of non-negative measurable functions with domain E that is pointwise convergent A.E. to the function f on the domain E. Then

$$\int_E f \leq minlim \int_E f_n.$$

Proof. First of all, notice that - without loss of generality - we can work on the set $E' \subseteq E$ where $(f_n)_{n\in\mathbb{N}}$ pointwise converges to a f, $\mu(E-E')=0$, since the integral on a domain of zero measure is zero.

Let $h : E \to \mathbb{R}$ be a non-negative measurable function such that $supp(h) \subseteq E'$, $\mu(supp(h)) < \infty$, and $h \leq f$ su E'.

The integral $\int_E h = \int_{E'} h$ is well-defined.

Define, for every $n \in \mathbb{N}$, the function h_n as follows:

$$h_n = min\{h, f_n\}.$$

The functions h_n are limited, measurable and with support of finite measure, and, hence, the integrals $\int_E h_n = \int_{E'} h_n$ are well-defined.

Furthermore, since$h_n \leq h \leq f = \lim_{n\to\infty} f_n$, the sequence $(h_n)_{n\in\mathbb{N}}$ is pointwise convergent to the function f.

We claim that, by setting

$$M = sup\{h(x); x \in E\} = sup\{h(x); x \in supp(h)\},$$

it follows that

$$|h_n(x)| \leq M, \qquad \forall n \in \mathbb{N}, \qquad x \in supp(h) \subseteq E' \subseteq E$$

and, hence, the sequence $(h_n)_{n\in\mathbb{N}}$ satisfy the conditions of Prop. 10 (dominated pointwise convergence). Then

$$\int_{supp(h)} h = \int_{E'} h = \int_E h = lim_{n\to\infty} \int_E h_n = minlim \int_E h_n \leq minlim \int_E f_n.$$

By considering the *sup* with respect to the functions of type h, we infer that

$$\int_E f \leq minlim \int_E f_n.$$

□

Remark 18. *In its general formulation, Fatou's Lemma is "the best possible result": in fact, given a sequence punctually convergent to a non-negative measurable functions It is not, in general, true that the relative sequence of integrals is convergent (for this reason we speak of "minlim"), nor that equals the value of the limit (if it exists).*

Example 9.

- *Consider the sequence of characteristic functions* $\chi_{[n,+\infty[}$, $n \in \mathbb{N}$ *(with domain* $\mathbb{R}$*). We have:* $\int_{\mathbb{R}} \chi_{[n,+\infty[} = +\infty$*. On the other hand, the sequence is pointwise convergent to the identically zero function* $f \equiv 0$*. Then*

$$\int_{\mathbb{R}} f = 0 \neq lim_{n\to\infty} \int_{\mathbb{R}} \chi_{[n,+\infty[} = \infty.$$

- *Consider the sequence of functions* $(f_n)_{n\in\mathbb{N}}$ *(with domain* $\mathbb{R}$*), where*

$$f_n = \chi_{[n,n+1[},\ n\ pari, \qquad f_n \equiv 0,\ n\ dispari.$$

The sequence is pointwise convergent to the identically zero function $f \equiv 0$*. However*

$$(\int_{\mathbb{R}} f_n)_{n\in\mathbb{N}} = (1, 0, 1, 0, \ldots),$$

that implies

$$\int_{\mathbb{R}} f = minlim \int_{\mathbb{R}} f_n = 0 < maxlim \int_{\mathbb{R}} f_n = 1.$$

Theorem 18. *(Beppo Levi , of the "monotone convergence") Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence of non-negative measurable functions with domain* E*.*

Assume that the sequence is non-decreasing monotone (that is, $f_n(x) \leq f_{n+1}(x),\ \forall x \in E,\ \ n \in \mathbb{N}.$*)*

By setting

$$f = lim_{n\to\infty} f_n,$$

we have

$$\int_E f = lim_{n\to\infty} \int_E f_n.$$

Proof. Notice that the monotonicity assumption

$$f_n(x) \leq f_{n+1}(x), \ \forall x \in E, \quad n \in \mathbb{N}$$

implies that

$$minlim f_n(x) = maxlim f_n(x) = lim_{n\to\infty} f_n(x) =^{def} f(x), \ \forall x \in E;$$

then the sequence $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent to the function $f = sup_n f_n$.
The Fatou Lemma implies that

$$\int_E f \leq minlim \int_E f_n. \qquad (\dagger)$$

On the other hand, for every $n \in \mathbb{N}$ we have $f_n \leq f$, and, hence, $\int_E f_n \leq \int_E f$; then

$$maxlim \int_E f_n \leq \int_E f. \qquad (\dagger\dagger)$$

Since, in general, $minlim \int_E f_n \leq maxlim \int_E f_n$, the conditions $(\dagger)$ e $(\dagger\dagger)$ imply

$$maxlim \int_E f_n = \int_E f = minlim \int_E f_n = lim_{n\to\infty} \int_E f_n.$$

$\square$

With the same arguments, the following variant of Beppo Levi's Theorem is immediately shown.

Proposition 14. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of non-negative measurable functions with domain E.*

Assume that the sequence is pointwise convergent to a function f and that

$$f_n \leq f, \qquad \forall n \in \mathbb{N}.$$

Then

$$\int_E f = lim_{n\to\infty} \int_E f_n.$$

A typical consequence of Beppo Levi's Theorem, very useful in applications, is provided by the following

Corollary 4. *Let $f : E \to \overline{\mathbb{R}}$ be a non-negative measurable function and let $(A_n)_{n\in\mathbb{N}},\ A_n \subseteq E$ be a nested non-decreasing sequence of measurable subsets of E, that is such that $A_n \subseteq A_{n+1},\ n \in \mathbb{N}$. Assume that*

$$\bigcup_{n\in\mathbb{N}} A_n = E.$$

Then

$$lim_{n\to\infty} \int_{A_n} f = \int_E f.$$

Proof. By construction, $(f \cdot \chi_{A_n})_{n\in\mathbb{N}}$ is a non-decreasing monotone sequence of non-negative measurable functions on E that is pointwise convergent to the function f.

From the Beppo Levi theorem, it follows

$$\int_E f = lim_{n\to\infty} \int_E f \cdot \chi_{A_n} = lim_{n\to\infty} \int_{A_n} f.$$

□

Example 10. *Consider the function $f\ :]0, +\infty + [\to \mathbb{R},\ f(x) = \frac{1}{x}$. The function f is non-negative, measurable (since it is continuous) and, hence, it is integrable on $]0, +\infty + [$. We compute $\int_{]0,+\infty+[} f$.*

Consider the nested sequence of closed intervals

$$([\frac{1}{n}, n])_{n\in\mathbb{Z}^+}$$

that satisfies the assumptions of the preceding Corollary.

Then

$$\int_{]0,+\infty+[} f = lim_{n\to\infty} \int_{[\frac{1}{n},n]} f.$$

on the interval $[\frac{1}{n}, n]$ the function f is also $R-$integrabile (since it is limited and continuous) and, furthermore,

$$\int_{[\frac{1}{n},n]} f = R\int_{\frac{1}{n}}^{n} f = [log(x)]_{x=\frac{1}{n}}^{x=n} = log(n) - log(\frac{1}{n}).$$

Then,

$$\int_{]0,+\infty+[} f = lim_{n\to\infty}(log(n) - log(\frac{1}{n})) = lim_{n\to\infty}log(n) - lim_{n\to\infty}log(\frac{1}{n}) = \infty - (-\infty) = \infty.$$

Example 11. *Fix a positive integer $m \in \mathbb{Z}^+$, and consider the function $f :]1, +\infty+[\to \mathbb{R}$, $f(x) = \frac{1}{x^m}$. The function f is non-negative and measurable.*
We compute the integral $\int_{]1,+\infty+[} f$.
Consider the nested sequence of closed intervals

$$([1 + \frac{1}{n}, n])_{n \in \mathbb{Z}^+}$$

that satisfies the assumptions of the preceding Corollary.
Then

$$\int_{]1,+\infty+[} f = lim_{n\to\infty} \int_{[1+\frac{1}{n},n]} f.$$

On every interval $[1 + \frac{1}{n}, n]$ the function f is R-integrable (since it is limited and continuous) and furthermoe

$$\int_{[1+\frac{1}{n},n]} f = R \int_{1+\frac{1}{n}}^{n} f.$$

We distingush two cases:

- *Let $m = 1$. Then*
$$\int_{]1,+\infty+[} f = lim_{n\to\infty}(log(n) - log(1 + \frac{1}{n})) =$$
$$lim_{n\to\infty} log(n) - lim_{n\to\infty} log(1 + \frac{1}{n}) = \infty - 0 = \infty.$$
- *Let $m > 1$. Then*
$$\int_{]1,+\infty+[} f = lim_{n\to\infty}[-(m-1)^{-1} \cdot x^{-m+1}]_{x=1+\frac{1}{n}}^{x=n} =$$
$$lim_{n\to\infty}(-(m-1)^{-1} \cdot n^{-m+1}) - lim_{n\to\infty}(-(m-1)^{-1} \cdot (1 + \frac{1}{n})^{m-1}) =$$
$$0 - (-(m-1)^{-1}) = (m-1)^{-1}.$$

Proposition 15. *(Beppo Levi for series)*
Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of non-negative measurable functions with domain E, and let

$$g = \sum_{n=0}^{\infty} f_n.$$

Then

$$\int_E g = \sum_{n=0}^{\infty} \int_E f_n.$$

Proof. Notice that the sequence of "partial sums"

$$(\sum_{k=0}^{n} f_k)_{n\in\mathbb{N}}$$

is non-decreasing monotone and that, by definition,

$$g = lim_{n\to\infty} \ (\sum_{k=0}^{n} f_k).$$

Since

$$\int_E \sum_{k=0}^{n} f_k = \sum_{k=0}^{n} \int_E f_k,$$

from the Beppo Levi theorem for sequences it follows:

$$\int_E g = lim_{n\to\infty} \ (\int_E \ \sum_{k=0}^{n} f_k) = lim_{n\to\infty} \ (\sum_{k=0}^{n} \ \int_E f_k) = \sum_{n=0}^{\infty} \ \int_E f_n.$$

□

Example 12. *Consider the function (on the domain* $[0, +\infty[$ *):*

$$g = \sum_{n=0}^{\infty} \ \frac{1}{n} \cdot \chi_{[n,n+1[}.$$

From the preceding result, we have:

$$\int_{[0,+\infty[} g = \sum_{n=0}^{\infty} \int_{[0,+\infty[} \frac{1}{n} \cdot \chi_{[n,n+1[} = \sum_{n=0}^{\infty} \frac{1}{n} = +\infty,$$

since the "harmonic series" of exponent 1 *is divergent.*

Definition 10. *A non-negative measurable function* $f : E \to \overline{\mathbb{R}}$ *sis said to be SUMMABLE (on E) if and only if* $\int_E f < \infty$.

Remark 19. *Beppo Levi's Theorem and its variants are a "typical" tool to check if a given function is summable or not.*

Example 13. *let* $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = e^{-x}, \ x \in \mathbb{R}$. *We wonder i* f *is summable on the halfline* $[0, +\infty[$ *or not:*

By Corollary 4*, setting* $f_n = f \cdot \chi_{[0,n]}, \ n \in \mathbb{Z}^+$*, we have:*

$$\int_{[0,+\infty[} f = lim_{n\to\infty} \int_{[0,+\infty[} f_n = lim_{n\to\infty} \int_{[0,n]} e^{-x} =$$

$$= lim_{n\to\infty}\ R\int_0^n e^{-x}dx = lim_{n\to\infty}[-e^{-x}]_{x=0}^{x=n} = lim_{n\to\infty}(-e^{-n}+1) = 1.$$

Hence f is summable on the halfline $[0,+\infty[$.

Example 14. *Consider the series of functions*

$$\sum_{n=1}^{\infty} \frac{1}{n}\cdot\chi_{]\frac{1}{n+1},\frac{1}{n}]}, \qquad (\diamondsuit)$$

on the domain $]0,1]$. The series $(\diamondsuit)$ pointwise converges on $]0,1]$ to a function g. Is g summable on $]0,1]$?

From the Beppo Levi theorem for series, we get

$$\int_{]0,1]} g = \sum_{n=1}^{\infty}\int_{]0,1]}\frac{1}{n}\cdot\chi_{]\frac{1}{n+1},\frac{1}{n}]} = \sum_{n=1}^{\infty}\frac{1}{n}\cdot\mu(]\frac{1}{n+1},\frac{1}{n}]) =$$

$$= \sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\frac{1}{n^3+n^2} < \sum_{n=1}^{\infty}\frac{1}{n^3} < \infty.$$

Indeed the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n^3}$ of exponent 3 is convergent, and, hence, we have

$$\int_{]0,1]} g < +\infty;$$

then g is summable on $]0,1]$.

Example 15. *Consider the series of functions*

$$\sum_{n=1}^{\infty} (n+1)\cdot\chi_{]\frac{1}{n+1},\frac{1}{n}]}, \qquad (\diamondsuit\diamondsuit)$$

on the domain $]0,1]$. The series $(\diamondsuit\diamondsuit)$ pointwise converges on $]0,1]$ to a function g. Is g summable on $]0,1]$?

From the Beppo Levi theorem for series, we get

$$\int_{]0,1]} g = \sum_{n=1}^{\infty}\int_{]0,1]}(n+1)\cdot\chi_{]\frac{1}{n+1},\frac{1}{n}]} = \sum_{n=1}^{\infty}(n+1)\cdot\mu(]\frac{1}{n+1},\frac{1}{n}]) =$$

$$= \sum_{n=1}^{\infty}(n+1)\cdot\frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\frac{1}{n} = \infty.$$

Indeed the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ of exponent 1 is divergent, and, hence, we have

$$\int_{]0,1]} g = +\infty;$$

then g is not summable on $]0,1]$.

2.6 Summable functions and the general Lebesgue integral

Let $f : E \to \overline{\mathbb{R}},\ E \subseteq \mathbb{R}^n$ be a function with extended real values.

Its *positive part* f^+ is the function (with the same domain E) defined as follows:

$$f^+(x) = f(x) \ se\ f(x) \geq 0, \qquad f^+(x) = 0 \ altrimenti.$$

Its *negative part* f^- is the function (with the same domain E) defined as follows:

$$f^-(x) = -f(x) \ se\ f(x) \leq 0, \qquad f^-(x) = 0 \ altrimenti.$$

Remark 20. *Let $f : E \to \overline{\mathbb{R}},\ E \subseteq \mathbb{R}^n$ be measurable.*

- *Its positive part f^+ and its negative part f^- are both non-negative measurable. Indeed, we have*

  $$f^+ = sup\ \{f,\ \mathbf{0}\}, \qquad\qquad f^- = sup\ \{-f,\ \mathbf{0}\}$$

  where $\mathbf{0}$ denotes the identically zero, that is clearly measurable.

- $$f = f^+ - f^-, \qquad\qquad |f| = f^+ - f^-.$$

Definition 11. • *A measurable function f is said to be SUMMABLE on E if and only if the functions $f^+,\ f^-$ are both summable on E, as non-negative functions.*

- *Iff is summable on E, we set, by definition:*

  $$\int_E f = \int_E f^+ - \int_E f^-.$$

Proposition 16. *Let f, g be summable functions on$E$. Then:*

- $$\int_E c \cdot f = c \cdot \int_E f, \qquad \forall c \in \overline{\mathbb{R}}.$$

- $$\int_E (f+g) = \int_E f + \int_E g.$$

- *se* $f \leq g$ *A.E., then*
$$\int_E f \leq \int_E g.$$
- *If* $A, B \subseteq E$ *are measurable, and* $A \cap B = \emptyset$*, then*
$$\int_{A \cup B} f = \int_A f + \int_B f.$$

.

Theorem 19. *(Lebesgue dominated convergence)*
Let g *be a summable function on* E *and let* $(f_n)_{n \in \mathbb{N}}$ *be a sequence of measurable functions on* E *such that:*

- *the sequence* $(f_n)_{n \in \mathbb{N}}$ *is pointwise convergent A.E. to the function* f.
- $|f_n| \leq g$ *on* E *(dominance condition).*

Then f *e' is summable on* E *and, furthermore,*
$$\int_E f = lim_{n \to \infty} \int_E f_n.$$

Proof. Notice that, from the dominance hypotesis, it immediately follows that
$$f^+ + f^- = |f| \leq g \ A.E.;$$
from Proposition 14, we infer
$$\int_E f^+ + \int_E f^- = \int_E |f| \leq \int_E g < +\infty,$$
and, hence, f^+, f^- are both summable (as non-negative functions), and then f is summable.
The functions $(g - f_n), n \in \mathbb{N}$ are non-negative measurable and the sequence $\Big((g - f_n)\Big)_{n \in \mathbb{N}}$ is pointwise convergent A.E. to the function $(g - f)$.
From Fatou Lemma, it follows that
$$\int_E (g - f) \leq minlim \int_E (g - f_n).$$

Now

$$\int_E g - \int_E f = \int_E (g-f) \leq minlim \ \int_E (g-f_n) = sup_{\ n\in\mathbb{N}} \left(inf_{\ k\geq n} \ \int_E (g-f_k) \right) =$$

$$= sup_{\ n\in\mathbb{N}} \left(\int_E g - \ sup_{\ k\geq n} \ \int_E f_k \right) =$$

$$= \int_E g - \ inf_{\ n\in\mathbb{N}} \left(sup_{\ k\geq n} \ \int_E f_k \right) = \int_E g - \ maxlim \ \int_E f_n.$$

Thus, we have proved

$$\int_E g - \int_E f \leq \int_E g - \ maxlim \ \int_E f_n,$$

and, then:

$$\int_E f \geq maxlim \ \int_E f_n.$$

Analogously, the functions $(g + f_n), n \in \mathbb{N}$ are non-negative measurable (recall that, by hypotesis, $g \geq |f_n|$) and the sequence $\Big((g + f_n)\Big)_{n\in\mathbb{N}}$ is pointwise convergent A.E. to the function $(g + f)$,
From Fatou Lemma, it follows that

$$\int_E (g+f) \leq minlim \ \int_E (g+f_n).$$

Now

$$\int_E g + \int_E f = \int_E (g+f) \leq minlim \ \int_E (g+f_n) = sup_{\ n\in\mathbb{N}} \left(inf_{\ k\geq n} \ \int_E (g+f_k) \right) =$$

$$= sup_{\ n\in\mathbb{N}} \left(\int_E g + inf_{\ k\geq n} \ \int_E f_k \right) =$$

$$= \int_E g + sup_{\ n\in\mathbb{N}} \left(\ inf_{\ k\geq n} \ \int_E f_k \right) = \int_E g + minlim \ \int_E f_n.$$

Thus, we have proved

$$\int_E g + \int_E f \leq \int_E g + \ minlim \ \int_E f_n,$$

and, then:

$$\int_E f \leq minlim \ \int_E f_n.$$

since, in general, "$minlim \leq maxlim$", we get

$$maxlim \int_E f_n = \int_E f = minlim \int_E f_n = lim_{n\to\infty} \int_E f_n.$$

$\square$

3 Computation of measures and integrals in domains in $\mathbb{R}^n$: the Fubini-Tonelli Theorem and " multiple integrals"

Recall that the measure of Lebesgue is always defined relative to the choice of a fixed space $\mathbb{R}^n$, that is, for a fixed finite dimension $n \in \mathbb{Z}^+$.

In the following, to highlight this aspect, we will write μ_n instead of μ.

We will use further notational conventions.

Given $n, r, s \in \mathbb{Z}^+, \quad n = r + s$ and a point $\alpha = (x_1, \ldots, x_r, y_1, \ldots, y_s) \in \mathbb{R}^n$, we will write $\alpha = (x, y)$, being $x = (x_1, \ldots, x_r) \in \mathbb{R}^r, \quad y = (y_1, \ldots, y_s) \in \mathbb{R}^s$.

In other words, by representing $\mathbb{R}^n \equiv \mathbb{R}^r \times \mathbb{R}^s$, a point $\alpha \in \mathbb{R}^n$ may be canonically regarded as an ordered pair (x, y), where $x = (x_1, \ldots, x_r) \in \mathbb{R}^r, \quad y = (y_1, \ldots, y_s) \in \mathbb{R}^s$.

Theorem 20. *(***Fubini – Tonelli***)*

Let $A \subseteq \mathbb{R}^n, \ n \geq 2, \ A$ be measurable.

Let $r, s \in \mathbb{Z}^+, \quad n = r + s$.

Let denote by (x, y) the generic point in $\mathbb{R}^n \equiv \mathbb{R}^r \times \mathbb{R}^s$, where $x = (x_1, \ldots, x_r) \in \mathbb{R}^r, \quad y = (y_1, \ldots, y_s) \in \mathbb{R}^s$.

Given a fixed $\underline{x} \in \mathbb{R}^r$, set

$$A_{\underline{x}} = \{y \in \mathbb{R}^s; \ (\underline{x}, y) \in A\} \subseteq \mathbb{R}^s,$$

and

$$S_A = \{\underline{x} \in \mathbb{R}^r; \ \mu_s^*(A_{\underline{x}}) > 0\} \subseteq \mathbb{R}^r.$$

The set

$$S_A = \{\underline{x} \in \mathbb{R}^r; \ \mu_s^*(A_{\underline{x}}) > 0\} \subseteq \mathbb{R}^r \textit{ is measurable.}$$

Let

$$f : A \to \overline{\mathbb{R}}$$

be a non-negative measurable function. Then (Tonelli)

1. *There exists $S_A^0 \subseteq S_A$ measurable with $\mu_r(S_A^0) = 0$ such that, for every $\underline{x} \in S_A - S_A^0$, the function $f_{\underline{x}} : A_{\underline{x}} \to \overline{\mathbb{R}}$ defined as*

   $$f_{\underline{x}}(y) = f(\underline{x}, y), \qquad y \in A_{\underline{x}}$$

   is a non-negative measurable function.

Thus, in particular,

$$A_{\underline{x}} = \{y \in \mathbb{R}^s;\ (x, y) \in A\} \subseteq \mathbb{R}^s \ e' \ misurabile \ \forall \underline{x} \in S_A - S_A^0.$$

(In plain words, the "sections" $A_{\underline{x}}$ are measurable A.E. on the set S_A).

2. *The function*

$$f_1 : \underline{x} \mapsto f_1(\underline{x}) = \int_{A_{\underline{x}}} f_{\underline{x}} = \int_{A_{\underline{x}}} f_{\underline{x}}(y)dy, \qquad \underline{x} \in S_A - S_A^0$$

(defined on the measurable set $S_A - S_A^0$) isnon-negative measurable.

3. *We have::*

$$\int_A f = \int_{S_A - S_A^0} f_1(\underline{x})d\underline{x} = \int_{S_A - S_A^0} (\int_{A_{\underline{x}}} f_{\underline{x}}(y)dy)d\underline{x}.$$

let

$$f : A \to \overline{\mathbb{R}}$$

be any measurable function, f summable onA. Then (Fubini) the functions $f_{\underline{x}}, f_1$ defined at 1) *e* 2) *are summable and the integral satisfies the identity above.*

Keeping the notations of the preceding Theorem, we have, as a special case:

Corollary 5. *(Tonelli Theorem for measures)*
Let $B \subseteq \mathbb{R}^n \equiv \mathbb{R}^r \times \mathbb{R}^s$, B measurable.
Set

$$S_B = \{\underline{x} \in \mathbb{R}^r; \mu_s^*(B_{\underline{x}}) > 0\}, \qquad S_B^0 = \{\underline{x} \in \mathbb{R}^r; B_{\underline{x}} \ NON \ misurabile\} \subseteq S_B.$$

1. *The subsetsS_B, $S_B^0 \subseteq \mathbb{R}^r$ are measurable, and $\mu_r(S_B^0) = 0$.*
2. *We have:*

$$\mu_n(B) = \int_{S_B - S_B^0} \mu_s(B_{\underline{x}}).$$

Proof. As an example / exercise, we derive this statement (by specialization) from the points 1), 2), 3) of the Theorem 20.

By following the notation above, let

$$A = \mathbb{R}^n,$$

thus

$$S_A = S_{\mathbb{R}^n} = \mathbb{R}^r$$

and

$$A_{\underline{x}} = {\mathbb{R}^n}_{\underline{x}} = \mathbb{R}^s, \qquad \forall \underline{x} \in \mathbb{R}^r.$$

Consider th *characteristic functions* $\chi_B : \mathbb{R}^n \to \mathbb{R}$, which is a non-negative measurable function, and therefore satisfies the conditions of Tonelli Theorem.

Form point 1), we know that there exists a subset $(\mathbb{R}^r)^0$ with ZERO MEASURE such that

$$(\chi_B)_{\underline{x}} \equiv \chi_{B_{\underline{x}}} : {\mathbb{R}^n}_{\underline{x}} = \mathbb{R}^s \to \mathbb{R}$$

is measurable for every $\underline{x} \in \mathbb{R}^r - (\mathbb{R}^r)^0$, that is $B_{\underline{x}}$ measurable, thus

$$(\mathbb{R}^r)^0 = S_B^0 \subseteq S_B.$$

from point 2), the function

$$(\chi_B)_1 : \underline{x} \mapsto \int_{\mathbb{R}^s} (\chi_B)_{\underline{x}} = \int_{\mathbb{R}^s} \chi_{B_{\underline{x}}} = \mu_s(B_{\underline{x}})$$

is measurable on $\mathbb{R}^r - (\mathbb{R}^r)^0 = \mathbb{R}^r - S_B^0$ (equivalently, on $\mathbb{R}^r$).

Then

$$supp((\chi_B)_1) = \{\underline{x} \in \mathbb{R}^r; B_{\underline{x}} \ misurabile, \ \mu_s(B_{\underline{x}}) > 0\}$$

is measurable, and

$$supp((\chi_B)_1) \cup S_B^0 = S_B$$

is, in turn, measurable.

Then, from point 3), it follows

$$\mu_n(B) = \int_{\mathbb{R}^n} \chi_B = \int_{\mathbb{R}^r - S_B^0} (\chi_B)_1 = \int_{S_B - S_B^0} (\chi_B)_1 = \int_{S_B - S_B^0} \mu_s(B_{\underline{x}}).$$

$\square$

Example 16. *1. Let*

$$A = \{(x,y) \in \mathbb{R}^2;\ x^2 + y^2 \le 1\},$$

and compute $\mu_2(A)$.

For $\underline{x} \in \mathbb{R}$, *we have*

$$A_{\underline{x}} = \emptyset, \qquad se\ |x| > 1,$$

$$A_{\underline{x}} = \{y \in \mathbb{R};\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\ \} \qquad se\ |x| \le 1,$$

the sets $A_{\underline{x}}$ *are closed, and thus measurable. Furthermore* $\mu(A_{\underline{x}}) = 2\sqrt{1-\underline{x}^2} > 0$, *for every* $\underline{x} \in]-1,1[$, *then* $S_A =]-1,1[$.

Therefore,

$$\mu_2(A) = \int_{S_A =]-1,1[} 2\sqrt{1-x^2} = R\int_{-1}^{1} 2\sqrt{1-x^2}dx.$$

2. *let*

$$A = \{(x,y) \in \mathbb{R}^2; x^2 < y \le 1\}$$

and compute $\mu_2(A)$.

Since' $A_{\underline{x}} = \{y \in \mathbb{R}; \underline{x}^2 < y \le 1\}$, *we have* $S_A =]-1,1[$.

Then

$$\mu_2(A) = \int_{]-1,1[} \mu_1(A_{\underline{x}}) = R\int_{-1}^{1} (1-x^2)dx = \big[x - \frac{x^3}{3}\big]_{x=-1}^{x=1} = \frac{4}{3},$$

and, hence,

$$\mu_2(A) = \frac{4}{3}.$$

3. *Let*

$$A = \{(x,y) \in \mathbb{R}^2; |x| < 1, |y| < 1\}, \qquad f : A \to \mathbb{R},\ f(x,y) = x + y,$$

and compute

$$\int_A f.$$

We have

$$S_A =]-1,1[, \qquad A_{\underline{x}} = \{y \in \mathbb{R}; -1 < y < 1\} =]-1,1[,\ \forall \underline{x} \in S_A.$$

Then

$$\int_A f = \int_{-1}^{1} f_1(x)dx$$

since, for every fixed $\underline{x} \in S_A =]-1, 1[$,

$$f_1(\underline{x}) = \int_{-1}^{1} (\underline{x} + y) dy = \big[\underline{x}y + \frac{y^2}{2}\big]_{y=-1}^{y=1} = 2\underline{x}.$$

Therefore,

$$\int_A f = \int_{-1}^{1} f_1(x) dx = \int_{-1}^{1} 2x dx = \big[x^2\big]_{x=-1}^{x=1} = 0.$$

4. Let

$$A = \{(x, y, z) \in \mathbb{R}^3;\ 0 \le z \le x^2 + y^2 + 2,\ |x| \le 1,\ |y| \le 1\}.$$

and compute

$$\mu_3(A).$$

We have $S_A =]-1, 1[$ *thus*

$$\mu_3(A) = \int_{-1 \le \underline{x} \le 1} \mu_2(A_{\underline{x}}) d\underline{x},$$

where, for every $\underline{x} \in]-1, 1[$, *we have*

$$A_{\underline{x}} = \{(y, z) \in \mathbb{R}^2;\ 0 \le z - y^2 \le \underline{x}^2 + 2,\ |y| \le 1\}.$$

Now

$$\mu_2(A_{\underline{x}}) = \int_{-1 \le \underline{y} \le 1} \mu_1((A_{\underline{x}})_{\underline{y}})\ d\underline{y},$$

where, for every $\underline{y} \in]-1, 1[$, *fixed*

$$(A_{\underline{x}})_{\underline{y}} = \{z \in \mathbb{R};\ 0 \le z \le \underline{x}^2 + \underline{y}^2 + 2\},$$

and thus

$$\mu_1((A_{\underline{x}})_{\underline{y}}) = \underline{x}^2 + \underline{y}^2 + 2.$$

Then,

$$\mu_2(A_{\underline{x}}) = \int_{-1 \le y \le 1} (\underline{x}^2 + y^2 + 2) dy = \big[(\underline{x}^2 + 2)y + \frac{y^3}{3}\big]_{y=-1}^{y=1} = 2(\underline{x}^2 + 2) + \frac{2}{3}.$$

Therefore,

$$\mu_3(A) = \int_{-1 \le x \le 1} \mu_2(A_x) dx = \int_{-1 \le x \le 1} (2(x^2 + 2) + \frac{2}{3}) dx =$$

$$= 2 \int_{-1 \le x \le 1} (x^2 + 2) dx + \frac{2}{3} \int_{-1 \le x \le 1} 1 dx = 2\big[\frac{x^3}{3} + 2x\big]_{x=-1}^{x=1} + \frac{4}{3} = \frac{32}{3}.$$

5. *From now on, to make the notation lighter, we will abandon the "underlined" notation and, therefore, we will write for example x in place of $\underline{x}$; from the reasoning phase, it should now be clear when x will point to a fixed or one point VARIABLE (on which to integrate).*

   Also, in subsequent sections, we will simply write $A_{x,y}$ instead of $(A_{\underline{x}})_{\underline{y}}$.

6. *Let*
$$A = \{(x, y, z) \in \mathbb{R}^3;\ 0 < z < -x^2 - y^2 + 1,\ |x| < 1,\ |y| < 1\}$$
   and compute
$$\mu_3(A).$$

   We have
$$S_A = \{(y, z) \in \mathbb{R}^2;\ \mu_2(A_x) > 0\} =]-1, 1[,$$
   then
$$\mu_3(A) = \int_{-1<x<1} \mu_2(A_x)dx.$$

   FIXED $x \in]-1, 1[$, to compute $\mu_2(A_x)$, we must consider, for every $y \in \mathbb{R}$, the "subsequent sections successive" $A_{x,y}$ (depending on x), and integrate $\mu_1(A_{x,y})$ on the set S_{A_x}.

   For every $x \in]-1, 1[$, we have
$$S_{A_x} = \{y \in \mathbb{R};\ -\sqrt{1-x^2} < y < \sqrt{1-x^2}\}.$$

   Then,
$$\mu_2(A_x) = \int_{-\sqrt{1-x^2}<y<\sqrt{1-x^2}} \mu_1(A_{x,y})dy,$$
   and, since
$$\mu_1(A_{x,y}) = -x^2 - y^2 + 1$$
   we have
$$\mu_2(A_x) = \int_{-\sqrt{1-x^2}<y<\sqrt{1-x^2}} (-x^2 - y^2 + 1)dy,$$
   and then
$$\mu_3(A) = \int_{-1<x<1} \Big(\int_{-\sqrt{1-x^2}<y<\sqrt{1-x^2}} (-x^2 - y^2 + 1)dy\Big)\ dx.$$

7. Let

$$B = \{(x, y) \in \mathbb{R}^2;\ |x| < 1,\ |y| < 1\}, \qquad f : B \to \mathbb{R}, \qquad f(x, y) = x^2 + y^2 + 2$$

and compute

$$\int_B f.$$

In terms of the "geometric" interpretation of the integral (being f non-negative), it is substantially immediate that

$$\int_B f = \mu_3(A),$$

since

$$A = \{(x, y, z) \in \mathbb{R}^3;\ 0 < z < x^2 + y^2 + 2,\ |x| < 1,\ |y| < 1\}$$

of Es. 4.

However, for the purpose of deepening / comparing, we re-calculate $\int_B f$ using the Fubini-Tonelli Theorem for integrals.

We have:

$$S_B = \{x \in \mathbb{R}; \mu_1(B_x) > 0\} =]-1, 1[, \qquad B_x = \{y \in \mathbb{R}; |y| \leq 1\}.$$

Then

$$f_1(x) = \int_{-1 \leq y \leq 1} (y^2 + (x^2 + 2))dy = \big[\frac{y^3}{3} + (x^2 + 2)y\big]_{y=-1}^{y=1} = \frac{2}{3} + 2(x^2 + 2).$$

It follows

$$\int_B f = \int_{S_B =]-1,1[} f_1(x)dx = \int_{-1}^{1} (2(x^2 + 2) + \frac{2}{3})dx =$$

$$= (2\big[\frac{x^3}{3} + 2x\big]_{x=-1}^{x=1} + \frac{4}{3} = \frac{32}{3} = \mu_3(A), \qquad (cfr.\ Es.\ 4).$$

8. *Let*

$$B = \{(x,y) \in \mathbb{R}^2;\ |x| < 1,\ |y| < 1\}, \qquad f : B \to \mathbb{R}, \qquad f(x,y) = -x^2 - y^2 + 1$$

and compute

$$\int_B f.$$

We have:

$$S_B = \{x \in \mathbb{R}; \mu_1(B_x) > 0\} = [-1,1], \qquad B_x = \{y \in \mathbb{R}; |y| \leq 1\}.$$

Then

$$f_1(x) = \int_{-1 \leq y \leq 1} (-y^2 + (-x^2+1))dy = \big[-\frac{y^3}{3} + (-x^2+1)y\big]_{y=-1}^{y=1} = -\frac{2}{3} + 2(-x^2+1).$$

It follows

$$\int_B f = \int_{S_B =]-1,1[} f_1(x)dx = \int_{-1}^{1} (2(-x^2+1) + \frac{2}{3})dx.$$

WE REMARK that

$$\int_B f \neq \mu_3(A),$$

where A is the set considered in Ex. 6.

Indeed, "geometrically" speaking, the graph of f

$$graph(f) = \{(x, y, -x^2 - y^2 + 1) \in \mathbb{R}^3\}$$

is the surface obtained by "rotating" the parabola of equation $z = -x^2 + 1$ in the XZ plane around to the Z axis, a surface that intersects the plane XY in the unit circle centered in the origin.

So $\mu_3(A)$ calculates the "volume" of the set of points in $\mathbb{R}^3$ between the XY plane and the surface $graph(f)$. In fact

$$A = \{(x,y,z) \in \mathbb{R}^3; x^2 + y^2 \leq 1,\ 0 \leq z \leq -(x^2+y^2) + 1\}.$$

On the other hand, the integral $\int_B f$ also computes, with NEGATIVE contribution, the "volume" of the set

$$A' = \{(x,y,z) \in \mathbb{R}^3; x^2 + y^2 > 1,\ |x| \leq 1,\ |y| \leq 1,\ 0 \geq z \geq -(x^2+y^2) + 1\}.$$

(This follows, for instance, from the fact that - by definition - $\int_B f = \int_B f^+ - \int_B f^-$.)

As a further exercise, calculate $mu_3(A')$ and verify that

$$\int_B f = \mu_3(A) - \mu_3(A').$$

9. *Let*

$$A=\{(x,y)\in\mathbb{R}^2; 0\le x\le 1,\ 0\le y\le 1\},\ f:A\to\mathbb{R},\ f(x,y)=\frac{x}{1+xy}.$$

and compute

$$\int_A f.$$

We have:

$$S_A=\{x\in\mathbb{R};\ \mu_1(A_x)>0\}=[0,1],\qquad A_x=\{y\in\mathbb{R};\ (x,y)\in A\}=[0,1]\ \forall x\in S_A.$$

Then

$$\int_A f=\int_{0\le x\le 1}\Big(\int_{0\le y\le 1}(\frac{x}{1+xy})dy\Big)dx=\int_{0\le x\le 1}\Big(\ \big[log(1+xy)\big]_{y=0}^{y=1}\Big)dx=$$

$$=\int_{0\le x\le 1} log(1+x)dx.$$

Using integration by part, we obtain

$$\int_A f=\big[x\cdot log(1+x)\big]_{x=0}^{x=1}-\int_{0\le x\le 1}\frac{x}{1+x}dx=log2-\big[x-log(1+x)\big]_{x=0}^{x=1}=2log2-1.$$

10. *Let*

$$A=\{(x,y)\in\mathbb{R}^2; 0\le x\le 2,\ 0\le y\le 1,\ y\le x^2\},\ f:A\to\mathbb{R},\ f(x,y)=x^2+y^2.$$

and compute

$$\int_A f.$$

We have

$$\int_A f=\int_{0\le x\le 2} f_1(x)dx=\int_{0\le x\le 1} f_1(x)dx+\int_{1<x\le 2} f_1(x)dx=$$

$$=\int_{0\le x\le 1} f_1(x)dx+\int_{1\le x\le 2} f_1(x)dx=$$

$$=\int_{0\le x\le 1}\Big(\int_{0\le y\le x^2}(x^2+y^2)dy\Big)dx+\int_{1\le x\le 2}\Big(\int_{0\le y\le 1}(x^2+y^2)dy\Big)dx=$$

$$=\int_{0\le x\le 1}\Big(\big[x^2y+\frac{y^3}{3}\big]_{y=0}^{y=x^2}\Big)dx+\int_{1\le x\le 2}\Big(\big[x^2y+\frac{y^3}{3}\big]_{y=0}^{y=1}\Big)dx=$$

$$=\int_{0\le x\le 1}(x^4+\frac{x^6}{3})dx+\int_{1\le x\le 2}(x^2+\frac{1}{3})dx=\Big[\frac{x^5}{5}+\frac{x^7}{21}\Big]_{x=0}^{x=1}+\Big[\frac{x^3}{3}+\frac{x}{3}\Big]_{x=1}^{x=2}=\frac{102}{35}.$$

11. We

$$A=\{(x,y)\in\mathbb{R}^2;\ x\geq y^4,\ y\geq x^2\},\ f:A\to\mathbb{R},\ f(x,y)=\sqrt{x}-y^2.$$

and compute

$$\int_A f.$$

We have

$$\int_A f=\int_{0\leq x\leq 1}\Big(\int_{x^2\leq y\leq x^{\frac{1}{4}}}(\sqrt{x}-y^2)dy\Big)dx=$$

$$=\int_{0\leq x\leq 1}\Big(\big[y\sqrt{x}-\frac{y^3}{3}\big]_{y=x^2}^{y=x^{\frac{1}{4}}}\Big)dx=$$

$$=\int_{0\leq x\leq 1}(x^{\frac{3}{4}}-\frac{1}{3}x^{\frac{3}{4}}-x^{\frac{5}{2}}+\frac{1}{3}x^6)dx=\int_{0\leq x\leq 1}(\frac{2}{3}x^{\frac{3}{4}}-x^{\frac{5}{2}}+\frac{1}{3}x^6)dx=$$

$$=\big[\frac{8}{21}x^{\frac{7}{4}}-\frac{2}{7}x^{\frac{7}{2}}+\frac{1}{21}x^7\big]_{x=0}^{x=1}=\frac{1}{7}.$$

12. Let

$$A=\{(x,y)\in\mathbb{R}^2;\ x\geq 1,\ y\geq 1,\ x+y\leq 3\},\ f:A\to\mathbb{R},\ f(x,y)=(x+y)^{-3}.$$

and compute

$$\int_A f.$$

We have

$$\int_A f=\int_{1\leq x\leq 2}\Big(\int_{1\leq y\leq 3-x}(x+y)^{-3}dy\Big)dx=\int_{1\leq x\leq 2}\Big(\big[-\frac{1}{2}(x+y)^{-2}\big]_{y=1}^{y=3-x}\Big)dx=$$

$$=-\frac{1}{2}\int_{1\leq x\leq 2}\big((x+3-x)^{-2}-(x+1)^{-2}\big)dx=-\frac{1}{2}\int_{1\leq x\leq 2}\big((\frac{1}{9}-(x+1)^{-2}\big)dx=$$

$$=-\frac{1}{18}+\frac{1}{2}\int_{1\leq x\leq 2}(x+1)^{-2}dx=-\frac{1}{18}-\frac{1}{2}\big[(x+1)^{-1}\big]_{x=1}^{x=2}=\frac{1}{36}.$$

13. Let

$$A=\{(x,y)\in\mathbb{R}^2;\ \frac{y^2}{4}\le x\le 4\},\ f:A\to\mathbb{R},\ f(x,y)=x^2-xy.$$

and compute

$$\int_A f.$$

We have

$$\int_A f=\int_{0\le x\le 4}\Big(\int_{-2\sqrt{2}\le y\le 2\sqrt{2}}(x^2-xy)dy\Big)dx=$$

$$=\int_{0\le x\le 4}\Big(\big[x^2y-\frac{xy^2}{2}\big]_{y=-2\sqrt{2}}^{y=2\sqrt{2}}\Big)dx=4\sqrt{2}\int_{0\le x\le 4}x^2dx=4\sqrt{2}\big[\frac{x^3}{3}\big]_{x=0}^{x=4}=4\sqrt{2}\frac{64}{3}.$$

14. Let

$$A=\{(x,y)\in\mathbb{R}^2;\ |x|+|y|\le 1\},\ f:A\to\mathbb{R},\ f(x,y)=|x|+|y|.$$

and compute

$$\int_A f.$$

By SYMMETRY both of the domain A (Why?), and of the function f (Why?), we have:

$$\int_A f=4\cdot\int_B (x+y)dy\ dx,$$

where

$$B=\{(x,y)\in\mathbb{R}^2;\ x\ge 0,\ y\ge 0,\ x+y\le 1\}.$$

Now

$$\int_A f=4\cdot\int_B (x+y)dy\ dx=\int_{0\le x\le 1}\Big(\int_{0\le y\le 1-x}(x+y)dy\Big)dx=$$

$$=\int_{0\le x\le 1}\Big(\big[xy+\frac{y^2}{2}\big]_{y=0}^{y=1-x}\Big)dx==\frac{1}{2}\int_{0\le x\le 1}(1-x^2)dx=\frac{1}{2}\big[x-\frac{x^3}{3}\big]_{x=0}^{x=1}=\frac{1}{3}.$$

15. *Let*
$$B = \{(x, y, z) \in \mathbb{R}^3;\ 0 \le x \le 1, 0 \le y \le 1, z \ge 0, x + y + z \le 2\}$$
and compute
$$\mu_3(B).$$

We have
$$\mu_3(B) = \int_{0 \le x \le 1} \mu_2(B_x) dx = \int_{0 \le x \le 1} \Big(\int_{0 \le y \le 1} \mu_1((B_x)_y) dy \Big) dx,$$
where
$$B_x = \{(y, z) \in \mathbb{R}^2;\ 0 \le y \le 1, z \ge 0, y + z \le 2 - x\} \qquad \forall x \in [0, 1]$$
e
$$(B_x)_y = \{z \in \mathbb{R};\ z \ge 0, z \le 2 - x - y\} \qquad \forall y \in [0, 1].$$
Since $\mu_1((B_x)_y) = 2 - x - y$, *we infer*
$$\mu_3(B) = \int_{0 \le x \le 1} \Big(\int_{0 \le y \le 1} (2 - x - y) dy \Big) dx = \int_{0 \le x \le 1} \Big(\big[2y - xy - \frac{y^2}{2}\big]_{y=0}^{y=1} \Big) dx =$$
$$= \int_{0 \le x \le 1} (\frac{3}{2} - x) dx = \Big[\ \frac{3}{2}x - \frac{x^2}{2}\ \Big] dx = 1.$$

16. *Fixed* $\rho \in \mathbb{R}^+$, *let*
$$B_\rho = \{(x, y, z) \in \mathbb{R}^3;\ x^2 + y^2 + z^2 \le \rho\}$$
and compute
$$\mu_3(B_\rho).$$

We have
$$(B_\rho)_x = \{(y, z) \in \mathbb{R}^2;\ y^2 + z^2 \le \rho - x^2\},$$
then
$$\mu_3(B_\rho) = \int_{-\sqrt{\rho} \le x \le \sqrt{\rho}} \mu_2((B_\rho)_x)\ dx = \int_{-\sqrt{\rho} \le x \le \sqrt{\rho}} \pi(\rho - x^2)\ dx =$$
$$= \pi\Big[\ \rho x - \frac{x^3}{3}\ \Big]_{x=-\sqrt{\rho}}^{x=\sqrt{\rho}} == \pi(\rho\sqrt{\rho} + \rho\sqrt{\rho} - \frac{\rho\sqrt{\rho}}{3} - \frac{\rho\sqrt{\rho}}{3}) = \frac{4}{3}\ \pi(\sqrt{\rho})^3.$$

17. Let

$$A = \{(x, y, z) \in \mathbb{R}^3;\ x^2 + y^2 - z^2 \leq 1,\ |z| \leq 1\}$$

and compute

$$\mu_3(A).$$

For every $z \in \mathbb{R},\ |z| \leq 1$, we have

$$A_z = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 \leq 1 + z^2\}$$

and then

$$\mu_2(A_z) = \pi(1 + z^2).$$

Hence,

$$\mu_3(A) = \int_{-1 \leq z \leq 1} \mu_2(A_z) dz = \int_{-1 \leq z \leq 1} (\pi(1 + z^2)) dz =$$

$$= \pi\big[z + \frac{z^3}{3}\big]_{z=-1}^{z=1} = \pi(2 + \frac{2}{3}) = \frac{8}{3}\pi.$$

18. *let* $p(x,y) = x^2y^2 - y^2 + x^2y - y \in \mathbb{R}[x,y]$*, consider the set*

$$A = \{(x,y) \in \mathbb{R}^2;\ p(x,y) = 0\}$$

and compute

$$\mu_2(A).$$

For simplicity of presentation, let's premise a couple of "geometric" observations:

- $$p(x,y) = x^2y^2 - y^2 + x^2y - y = (x^2-1)y^2 + (x^2-1)y = (x^2-1)(y^2+y).$$
- *Thus,* A *is the union of four lines in the plane* $\mathbb{R}^2$*:*
  two vertical lines
  $$\{(x,y) \in \mathbb{R}^2;\ x = -1\} \qquad \{(x,y) \in \mathbb{R}^2;\ x = 1\},$$
  and two horizontal lines
  $$\{(x,y) \in \mathbb{R}^2;\ y = 0\} \qquad \{(x,y) \in \mathbb{R}^2;\ y = -1\}.$$

Following the notation of the 18) *point, we have the following description of the "sections"*

$$A_{\underline{x}}, \qquad \underline{x} \in \mathbb{R}.$$

- *If* $\underline{x} = \pm 1$*, then*
  $$A_{\underline{x}} = \{(x,y) \in \mathbb{R}^2;\ p(\underline{x},y) = 0\} = \mathbb{R},$$
  where $p(\underline{x},y) \in \mathbb{R}[y]$ *is the zero polynomial in the variable* y*.*
  Then, if $\underline{x} = \pm 1$*, and, hence,* $\mu_1(A_{\underline{x}}) = \infty > 0$*.*
- *If* $\underline{x} \neq \pm 1$*, then*
  $$A_{\underline{x}} = \{(x,y) \in \mathbb{R}^2;\ p(\underline{x},y) = 0\} = \{0,-1\}.$$
  Then, se $\underline{x} \neq \pm 1$*, and, hence,* $\mu_1(A_{\underline{x}}) = 0$*.*

it follows that the set

$$S_A = \{\underline{x} \in \mathbb{R};\ \mu_1(A_{\underline{x}}) > 0\} = \{-1,1\}$$

is such that

$$\mu_1(S_A) = \mu_1(\{-1,1\}) = 0.$$

Therefore

$$\mu_2(A) = \int_{S_A=\{-1,1\}} \mu_1(A_{\underline{x}})d\underline{x} = 0.$$

19. *Let*
$$p(x,y,z) = xyz^2 - xz^2 + x^2y^2z - x^2z + xy - x \in \mathbb{R}[x,y,z],$$
and consider the set
$$A = \{(x,y,z) \in \mathbb{R}^3;\ p(x,y,z) = 0\}.$$
Prove that
$$\mu_3(A) = 0.$$
Notice that
$$p(x,y,z) = (xy-x)z^2 + (x^2y^2 - x^2)z + (xy - x).$$
Let
$$a_2(x,y) = xy - x, \quad a_1(x,y) = x^2y^2 - x^2, \quad a_1(x,y) = xy - x.$$
Given $(\underline{x}, \underline{y}) \in \mathbb{R}^2$, *the "section"*
$$A_{(\underline{x},\underline{y})} = \{z \in \mathbb{R};\ p(\underline{x}, \underline{y}, z) = 0\}$$
is such that
$$\mu_1(A_{(\underline{x},\underline{y})}) > 0$$
if and only if the polynomial $p(\underline{x}, \underline{y}, z) \in \mathbb{R}[z]$ *is the zero polynomial in the variable* z.

Then, the set
$$S_A = \{(\underline{x}, \underline{y}) \in \mathbb{R}^2;\ \mu_1(A_{(\underline{x},\underline{y})}) > 0\} =$$
$$= \{(\underline{x}, \underline{y}) \in \mathbb{R}^2;\ a_2(\underline{x}, \underline{y}) = 0, a_1(\underline{x}, \underline{y}) = 0, a_0(\underline{x}, \underline{y}) = 0\}$$
is such that
$$\mu_2(S_A) = 0.$$
It follows
$$\mu_3(A) = \int_{S_A} \mu_1(A_{(\underline{x},\underline{y})}) d\underline{x} d\underline{y} = 0.$$

20. *Let*
$$A = \{(x,y,z) \in \mathbb{R}^3;\ x^2 - y^2 - z^2 > 1,\ |x| > 1,\ |x| \le 2\},$$
and compute
$$\mu_3(A).$$
To foster "geometric" intuition, we observe that the set

$$V = \{(x, y, z) \in \mathbb{R}^3;\ x^2 - y^2 - z^2 = 1\}$$

is the surface ("two-sided hyperboloid") obtained by rotating (in the space $\mathbb{R}^3$) the hyperbola of equation $x^2 - y^2 = 1$ in the XY plane around the X. axis

For reasons of symmetry, we have therefore

$$\mu_3(A) = 2\mu_3(B),$$

where

$$B = \{(x, y, z) \in \mathbb{R}^3;\ x^2 - y^2 - z^2 > 1,\ 1 < x \leq 2\}.$$

Given $x \in \mathbb{R},\ 1 < x \leq 2,$ we have

$$B_x = \{(y, z) \in \mathbb{R}^2;\ y^2 + z^2 \leq x^2 - 1\},$$

then

$$\mu_2(B_x) = \pi(x^2 - 1).$$

Therefore

$$\mu_3(B) = \int_{1<x\leq 2} \mu_2(B_x)dx = \int_1^2 (\pi(x^2 - 1))dx = \pi\Big[\frac{x^3}{3} - x\Big]_{x=1}^{x=2} = \frac{4}{3}\pi,$$

and thus

$$\mu_3(A) = \frac{8}{3}\pi.$$