A few introductory words on the contents of this course.

First of all, the real name of the course should have been LEBESGUE MEASURE THEORY AND INGRATION THEORY, as the course will be divided into two parts, one preliminary to the other, specifically, Measure Theory and subsequently Integration Theory.

From a historical point of view, Lebesgue's measurement and integration theory were born in the context of Real Analysis, especially integration theory.

As is well-known, differential/integral calculus originates from the pioneering and independent work of Newton and Leibniz in the 18th century, obtained a first rigorous formulation at the beginning of the 19th century essentially thanks to Augustin Cauchy, and a finally definitive formulation by Bernhard Riemann around the middle of the 19th century.

However, one - in a certain sense - limitation of the theory immediately became clear: the class of integrable functions in the Rieman sense (Riemann integrable functions) seems to be rather restricted, that is, the class

of more or less continuous functions. Many efforts were made, in the second half of the 19th century, with the aim, for example, of obtaining a necessary and sufficient condition for the integration of a function of a variable on an interval of the real line, but without success.

See for example the beautiful historical appendices in the monographs Mathematical Analysis by Enrico Giusti (Bollati Boringhieri). An explanation of this apparent failure will be given in this course.

This necessary and sufficient condition for Riemann integrability (nowadays known as the Lebesgue-Vitali theorem) requires precisely the Lebesgue measure theory, developed essentially by Henry Lebegue at the beginning of the 20th century.

The theory of Lebesgue integration (for functions of n arbitrary real variables) was introduced by the French mathematician Henry Lebesgue in his Doctoral Thesis at the beginning of the 20th century, and then also developed by other notable mathematicians such as Emile Borel, Giuseppe Vitali, Constantin Caratheodory, to name just a few.

However, Lebesgue's vision requires, as a prerequisite, a fundamental generalization of the notions of length, area, volume to "strange" subsets (in a certain sense arbitrary) of the real line, of the plane, of the space, etc.

To anticipate a simple but relevant example.

If we consider the closed interval [0,1] with endpoints 0 and 1, it is natural to define its measure as its length, that is 1.

But if we consider the set of points in the interval from 0 to 1 having RATIONAL coordinates, what will be its measure/length?

And what if we consider the set of points in the interval from 0 to 1 having an IRRATIONAL coordinate?

As we will see, the answers to these two questions will be clear from the Lebesgue Measure Theory: the first set (rational points) is of measure ZERO, the second set is of measure ONE.

This fact heralds notable links between Lebesgue Measure Theory and Probability Theory, which we will talk about shortly.

The first and fundamental characteristic of the Lebesgue integration theory is that the class of Lebesgue integrable functions (in any number of variables) is extremely broader than that of Riemann integrable functions: essentially, a function is Lebesgue Integrable if and only if it is a measurable function.

The class of measurable functions is an extremely broader class than the class of continuous functions almost everywhere (the class of integrable Riemann functions, from the Lebesgue-Vitali theorem). Furthermore, if a function is Riemann integrable, it is also Lebesgue integrable and the two integrals coincide.

Therefore, Lebesgue theory is a very strong extension of Riemann theory.

Other characteristics of the Lebesgue theory should also be mentioned. First of all, it is essentially dimension free: no difference between the case of one-variable functions and the case of several variables functions.

More relevant: the class of measurable functions is stable with respect to algebraic operations (sum, scalar multiplication, multiplication between functions) as is the class of continuous functions. But unlike the latter, it is also stable with respect to order operations, such as lower and upper bounds,

minimum and maximum limit and, consequently, pointwise limit.

Recall that the class of continuous functions is not stable, not even with respect to the pointwise limit. It is only with respect to the uniform limits (we will see this in detail during this course).

In the 1920s, this fact led numerous mathematicians (such as Hahn, Caratheodory, Frechet and especially Kologogorov) to understand that it provided the right and rigorous foundation

mathematician for Probability Theory. A measure of Probability

and simply a measure for which the measure of all space is equal to 1. Consequently, all the notions and results of Lebegue's measurement and integration theory have a probabilistic version, the theory is seen as the mathematical foundation/method of the Theory Of Probabilities. We will try, during the course, to explicitly mention these links.

Just to give a particularly relevant example, the notion of random variable is nothing other than the notion of measurable function!!!

(when the measure of space is equal to one, i.e. a measure of Probability).