Wednesday, February 14, 2024 11:14 AM

LIMITED OPEN INTERVALS IN $\mathbb{R}^n$

$a_1, \dots, a_n \in \mathbb{R}$, $b_1, \dots, b_n \in \mathbb{R}$ S.T.

$a_i < b_i$, $i = 1, 2, \dots, n$

CONSIDER THE SET

$$I = \{ (x_1, \dots, x_n) \in \mathbb{R}^n \ ; \ a_i < x_i < b_i \ , \ i = 1, 2, \dots, n \}$$

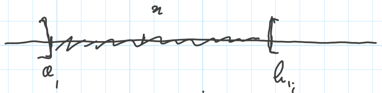
LIM. OPEN INTERVAL

EX $n = 1$

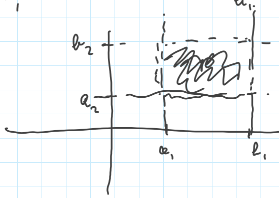
$$I = \{ x \in \mathbb{R} \ ; \ a_1 < x < b_1 \}$$

OPEN

$\mathbb{R}$



EX $n = 2$



$\mathbb{R}^2$

$$I = \{ (x_1, x_2) \in \mathbb{R}^2 \ ; \ a_i < x_i < b_i \ ; \ i = 1, 2 \}$$

$$I = \{ (x_1, \dots, x_n) \in \mathbb{R}^n \ ; \ a_i < x_i < b_i \ , \ i = 1, 2, \dots, n \}$$

MEASURE IS

$$\mu(I) \overset{\text{DEF}}{=} \prod_{i=1}^{n} \underset{\substack{\vee \\ 0}}{(b_i - a_i)} > 0 \quad !!!$$

LET $A \subseteq \mathbb{R}^n$.

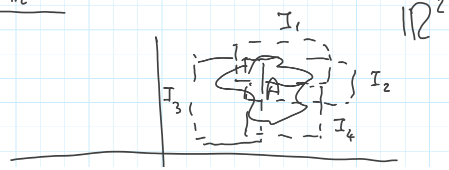
"LEBESGUE" COVERING OF $A \subseteq \mathbb{R}^n$ IS BY DEF

$(\ast)$ $\{ I_k \ ; \ k \in \Omega \}$ SUCH THAT

OPEN LIM. INTERVAL

i) $A \subseteq \bigcup_{k \in \Omega} I_k$ COVERING PROPERTY

EX IN $\mathbb{R}^2$



$A \subseteq I_1 \cup I_2 \cup I_3 \cup I_4$

ii) THE FAMILY OF INTERVALS

$\{ I_k \, ; \, k \in \mathbb{Q} \}$ IS <u>AT MOST COUNTABLE</u> !!!

<u>OUTER MEASURE</u>

LET $A \subseteq \mathbb{R}^n$. BY DEF

$$\rightarrow 0 \leq \mu^*(A) \overset{\text{DEF}}{=} \inf \Big\{ \sum_{k \in \mathbb{Q}} \mu(I_k) \, ; \text{ where } \{ I_k ; k \in \mathbb{Q} \} \text{ is } \underline{\text{a Leb. cov. of } A} \Big\}$$

OUTER MEAS.

$\mu(I_k) \geq 0$

<u>A FIRST QUESTION</u>: ARE THERE SUBSETS $A \subseteq \mathbb{R}^n$ SUCH THAT $\mu^*(A) = 0$????? <u>YES</u>

<u>ANSWER</u> (<u>EXAMPLE</u>)

$x \in \mathbb{R}^n$, AND $\{ x \} \subseteq \mathbb{R}^n$ x <u>arbitrary</u>

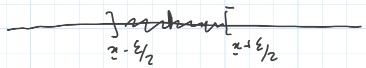
WHAT IS $\mu^*(\{ x \}) = ?$

FIX $\varepsilon \in \mathbb{R}^+$ ARBITRARY , AND CONSIDER

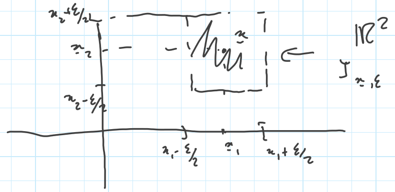
OPEN LIMITED INTERVAL $\rightarrow$ $I_{x,\varepsilon} := \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \, ; \, x_i - \varepsilon/2 < x_i < x_i + \varepsilon/2 \, , \, i = 1, 2, \ldots, n \}$

$x \in I_{x,\varepsilon}$

<u>EX</u> $n = 1$



<u>EX for $n = 2$</u>



NOW

<u>$\forall \varepsilon \in \mathbb{R}^+$</u> , $x \in I_{x,\varepsilon}$ $\Rightarrow$ $\{ I_{x,\varepsilon} \}$ IS A <u>LEB. COV.</u> OF THE SET $\{ x \}$ $(*)$

NOW . RECALL THAT

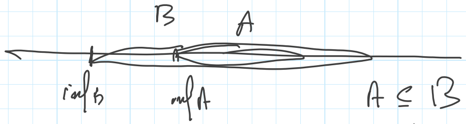
$\mu^*(\{ x \}) \overset{\text{DEF}}{=} \inf \{ \sum \mu(I_k) \, ; \text{ where } \{ I_k ; k \in \mathbb{Q} \} \text{ is}$,

[illegible] $\wedge$ U A LEB. COV OF $\{x\}$

$\inf\{\mu(I_{x,\varepsilon})\ ;\ \varepsilon \in \mathbb{R}^+\}$

IN PLAIN WORDS

$\mathbb{R}$



$A \subseteq B$

$\Downarrow$

$\inf B \leq \inf A$

SO, WE HAVE

$$0 \leq \mu^*(\{x\}) \leq \inf\{\mu(I_{x,\varepsilon})\ ;\ \varepsilon \in \mathbb{R}^+\}$$

$$\uparrow \varepsilon^n$$

$$\leq \inf\{\varepsilon^n\ ;\ \varepsilon \in \mathbb{R}^+\} = 0 !!!$$

THEREFORE, $\mu^*(\{x\}) = 0 !!!$

QUESTIONS ?

BREAK

BEGIN AGAIN AT ~ 17.05