Wednesday, February 14, 2024 11:16 AM

BEGIN AT 17.05

USEFUL NOTATION : LET $A \subseteq \mathbb{R}^n$

LET

$\mathcal{J}_A$ = THE FAMILY OF ALL LEBESGUE COVERINGS OF A

IN PARTICULAR , $A \subseteq \mathbb{R}^n$

$$\mu^*(A) \overset{DEF}{=} \inf \left\{ \sum_{k \in \alpha} \mu(I_k) \; ; \; \{I_k ; k \in \alpha\} \in \mathcal{J}_A \right\}.$$

α COUNT $\Rightarrow \sum_{k\in\alpha} = \sum_{n=0}^{\infty}$ SERIES

MAIN PROPERTIES OF THE OUTER MEASURE

PROP 1 (MONOTONICITY PROP.)

$$A \subseteq B \implies \mu^*(A) \leq \mu^*(B).$$

PROOF

$$\mu^*(A) \overset{DEF}{=} \inf \left\{ \sum_{k\in\alpha} \mu(I_k) \; ; \; \{I_k ; k\in\alpha\} \in \mathcal{J}_A \right\}$$

$$\wedge \qquad \cup \qquad \cup$$

$$\mu^*(B) \overset{DEF}{=} \inf \left\{ \sum_{k\in\alpha} \mu(I_k) \; ; \; \{I_k ; k\in\alpha\} \in \mathcal{J}_B \right\}.$$

BUT NOW , IF $A \subseteq B$ WHAT IS THE RELATION BETWEEN THE FAMILIES $\mathcal{J}_A$, $\mathcal{J}_B$?

$\Rightarrow \mathcal{J}_B \subseteq \mathcal{J}_A$!!!

QED !!!

COROLLARY $\mu^*(\emptyset) = 0$!!!

AS A MATTER OF FACT

$\emptyset \subseteq \{z\}$ $\quad z \in \mathbb{R}^n$ BUT

$$0 \leq \mu^*(\emptyset) \overset{MONOTON}{\leq} \mu^*(\{z\}) = 0 \implies \mu^*(\emptyset) = 0 \;!!!$$

PROP 2 (CONSISTENCY) $\quad I$ LIMITED OPEN INTERVAL IN $\mathbb{R}^n$

$$\mu^*(I) = \mu(I) = \mu^*(\overline{I})$$

WHERE $\overline{I}$ IS THE CLOSURE OF THE OPEN LIM. INTERVAL I.

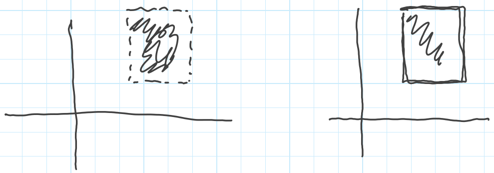
?

$I \subseteq \mathbb{R}^n$ ITS CLOSURE IS BY DEF

$$\overline{I} = \bigcap_{\substack{C \text{ CLOSED} \\ C \supseteq I}} C \qquad \text{CLOSED}, \ \overline{I} \supseteq I.$$

IN PLAIN WORDS $n=2$

I $\qquad\qquad$ $\overline{I}$



PROP. 3 LET $\{ A_k \subseteq \mathbb{R}^n ; \ k \in \mathcal{Q} \ \text{AT MOST COUNTABLE} \}$

THEN

$$\mu^*\Big(\bigcup_{k \in \mathcal{Q}} A_k \Big) \leq \sum_{k \in \mathcal{Q}} \mu^*(A_k) \quad !!!$$

EX $A_1, A_2 \subseteq \mathbb{R}^2$

$$\mu^*(A_1 \cup A_2) \leq \mu(A_1) + \mu^*(A_2)$$

COROLLARY $A \subseteq \mathbb{R}^n$, A IS AT MOST COUNTABLE.

THEN $\mu^*(A) = 0$!!!

WHY? IN GENERAL, IT IS TRUE

$$A = \bigcup_{a \in A} \{a\} \quad . \text{ BUT}$$

IF A COUNT $\Longrightarrow A = \overset{\text{COUNT.}}{\bigcup_{a \in A}} \{a\} \overset{\text{PROP. 3}}{\Longrightarrow}$

$$\mu^*(A) = \mu^*\Big(\overset{\text{COUNT}}{\bigcup_{a \in A}} \{a\} \Big) \overset{\text{PROP}}{\leq} \sum_{a \in A} \underbrace{\mu^*(\{a\})}_{= 0} = 0 !!! \qquad \text{QED}.$$

IN PARTICULAR, SINCE $\mathbb{Q} \subseteq \mathbb{R}$ IS COUNTABLE

(FIRST PARTIAL THM) $\Longrightarrow \mu^*(\mathbb{Q}) = 0$

WHY $\mu^*(\mathbb{R}) = +\infty$??? CLEARLY

FOR ANY $n \in \mathbb{Z}^+$,

(1) $]0,n[\subseteq \mathbb{R}$. BUT

$$\mu^*(]0,n[) = \mu(]0,n[) = n \underset{(1)}{\overset{\text{MONOTONICITY}}{\leq}} \mu^*(\mathbb{R})$$

BUT, HENCE

$$\mu^*(\mathbb{R}) \geq n \, , \, n \in \mathbb{Z}^+ \Rightarrow \mu^*(\mathbb{R}) = +\infty .$$

NOTICE THAT

$$\mu^*(\mathbb{Q}) = 0 \quad \text{AND} \quad \mu^*(\mathbb{R}) = +\infty . \quad (\circledast)$$

<u>FOR INSTANCE</u> $\mu^*(]0,1[) = \underline{1}$

$\mu^*(]0,1[\cap \mathbb{Q}) = 0$!!!

NO QUESTIONS? OK

BYE BYE, SEE YOU TOMORROW