Wednesday, February 14, 2024 11:17 AM

Hello, good morning!

Questions?

$\mathbb{Q} \subseteq \mathbb{R}$

$\uparrow$ RATIONAL NUMBERS

1) WHY $\mu^*(\mathbb{Q}) = 0$?

RECALL THAT, $\mathbb{Q}$ IS A COUNTABLE SET.

NOW

$$\underset{\text{COUNT.}}{\mathbb{Q}} = \overset{\text{COUNT. !!!}}{\bigcup_{q \in \mathbb{Q}}} \{q\}$$ THEREFORE, COUNT. SUBADD. PROP.

THAT IS : $\mu^*(\mathbb{Q}) = \mu^*\left(\overset{\text{COUNT}}{\bigcup_{q \in \mathbb{Q}}} \{q\}\right) \leq$

$$\leq \underset{\text{SERIES}}{\sum_{q \in \mathbb{Q}}} \underbrace{\mu^*(\{q\})}_{0} = 0 \text{ !!!}$$

HENCE $0 \leq \mu^*(\mathbb{Q}) \leq 0 \Rightarrow \mu^*(\mathbb{Q}) = 0$ QED

2) WHY $\mu^*(\mathbb{R}) = +\infty$???

A MATTER OF FACT, $\forall n \in \mathbb{Z}^+$ WE HAVE

$$\mu^*(]0,n[) = \mu(]0,n[) \overset{\text{RE}}{=} n - 0 = n \text{ !!!}$$

BUT NOW, FOR ANY $n \in \mathbb{Z}^+$

$$]0,n[\subseteq \mathbb{R} \overset{\text{MONOTONICITY}}{\Longrightarrow} n = \mu^*(]0,n[) \leq \mu^*(\mathbb{R})$$

$$\Rightarrow \mu^*(\mathbb{R}) \geq n \quad \forall n \in \mathbb{Z}^+ \Rightarrow \mu^*(\mathbb{R}) = +\infty \text{ !!!}$$

VARIATION ON THE THEM !

1) $\mu^*(]0,1[) = \mu(]0,1[) = 1$

2) $\mu^*(]0,1[\cap \mathbb{Q}) \overset{\text{MONOT.}}{\leq} \mu^*(\mathbb{Q}) = 0$!!

$\downarrow$ THIS IS 0.

PROP 3 (COUNT. SUBADDITIVITY)

LET $\{A_k \subseteq \mathbb{R}^n ; k \in \mathcal{A}\}$ AN AT MOST COUNTABLE

$\mathbb{R}^n$ THEN

FAMILY OF SUBSETS IN $\mathbb{R}$. THEN

$$\mu^*\left(\bigcup_{k\in a} A_k\right) \leq \sum_{k\in a} \mu^*(A_k) \quad !!!$$

PROOF FIRST, WE NOTICE THAT,

FOR ANY FIXED A_k THE FOLLOWING IS TRUE:

(*) $\forall \varepsilon \in \mathbb{R}^+ \quad \exists \{I_{k_j} ; j \in a_k\} \in \mathfrak{Z}_{A_k}$ $\boxed{\text{IS TRUE}}$

intervals

SUCH THAT

$$\sum_{j\in a_k} \mu(I_{k_j}) \leq \mu^*(A_k) + \varepsilon/2^k \quad !!!$$

WHY (*) IS TRUE ??? BY DEFINITION !!! BY CONTRADICTION

SUPPOSE (*) IS FALSE $\Longrightarrow$

(**) $\exists \varepsilon \in \mathbb{R}^+$ SUCH THAT FOR EVERY

$$\{I_{k_j} ; j \in a_k\} \in \mathfrak{Z}_{A_k}$$

WE HAVE

$$\sum_{j\in a_k} \mu(I_{k_j}) > \mu^*(A_k) + \varepsilon/2^k$$

TAKING THE INF HERE

$$\Downarrow$$

$$\mu^*(A_k) \geq \mu^*(A_k) + \varepsilon/2^k \quad \text{ABSURD!!!}$$

SO NOW, FOR ANY A_k IN OUR AT MOST COUNT FAMILY,

CHOOSE A LEB. COVERING SATISFIEDNG (*).

BUT NOW

$$\{I_{k_j} ; j \in a_k\} \in \mathfrak{Z}_{A_k}$$

AND CONSIDER

count.

$$\bigcup_{k\in a} \underbrace{\{I_{k_j} ; j\in a_k\}}_{\text{AT MOST COUNT}}$$ IS IT A LEBESGUE COVERING OF $\bigcup_{k\in a} A_k$???

THIS TRIVIALLY A COVERING OF $\bigcup_{k\in a} A_k$!!!

IT IS LEBESGUE DUE TO THE FIRST CANTOR THM!!!

SO BY DEF

$$\mu^*\left(\bigcup_{k\in a} A_k\right) \leq$$

$$\leq \sum_{\substack{j\in a_k \\ k\in a}} \mu(I_{k_j}) \quad !!!$$

THEN

$$\mu^*\Big(\bigcup_{k\in\mathcal{A}} A_k\Big) \leq \sum_{k\in\mathcal{A}} \Big(\sum_{j\in\mathbb{R}_k} \mu(I_{k_j})\Big) \leq$$

TRIVIAL FACT

$$\leq \sum_{k\in\mathcal{A}} \Big(\mu^*(A_k) + \varepsilon/2^k\Big) =$$

$$= \sum_{k\in\mathcal{A}} \mu^*(A_k) + \sum_{k\in\mathcal{A}} \varepsilon/2^k = \sum_{k\in\mathcal{A}} \mu^*(A_k) + \varepsilon$$

IN THE WORST CASE ($\mathcal{A}$ INFINITE BUT COUNTABLE)

$$\sum_{k\in\mathcal{A}} \frac{\varepsilon}{2^k} = \sum_{k=1}^{\infty} \varepsilon/2^k = \varepsilon \cdot \sum_{k=1}^{\infty} \frac{1}{2^k} = \varepsilon \cdot 1$$

GEOMETRIC SERIE OF REASON $1/2$

SO, WE PROVED THE (INFINITE) FAMILY OF INEQUALITIES

$$(\dagger) \qquad \mu^*\Big(\bigcup_{k\in\mathcal{A}} A_k\Big) \leq \sum_{k\in\mathcal{A}} \mu^*(A_k) + \varepsilon \quad , \forall\, \varepsilon \in \mathbb{R}^+$$

$$\Downarrow$$

$$(\dagger\dagger) \qquad \mu^*\Big(\bigcup_{k\in\mathcal{A}} A_k\Big) \leq \sum_{k\in\mathcal{A}} \mu^*(A_k)$$

QED ⊠

BREAK WE BEGIN 10.20