Wednesday, February 14, 2024 11:18 AM

EX / APPL LET $\underline{A = \{(x,y) \in \mathbb{R}^2 ; y \in \mathbb{Q}\}}$

WHAT IS $\mu^*(A)$???

NOTICE THAT

$$A = \{(x,y) \in \mathbb{R}^2 ; y \in \mathbb{Q}\} =$$

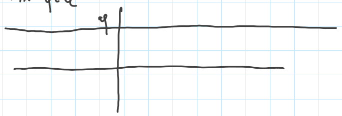
$$= \bigcup_{q \in \mathbb{Q}}^{\text{COUNT}} \{(x,y) \in \mathbb{R}^2 ; y = q\} \implies$$

COUNT SUBADD PROP. IMPLIES THAT

$$\mu^*(A) \leq \sum_{q \in \mathbb{Q}}^{\text{COUNT}} \mu^*\left(\{(x,y) \in \mathbb{R}^2 ; y = q\}\right)$$

WHAT ARE THESE SETS

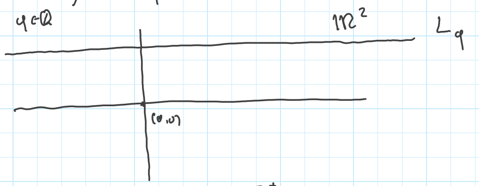
FIX $q \in \mathbb{Q}$



$L_q = \{(x,y) \in \mathbb{R}^2 ; y = q\}$

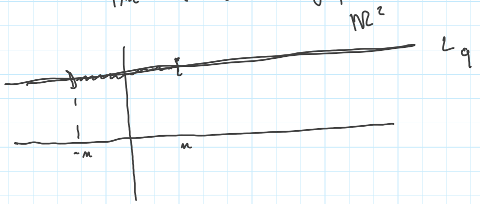
NOW, WHAT IS (FIXED $q \in \mathbb{Q}$)

$$\mu^*(L_q) \overset{\text{DEF}}{=} \mu^*\left(\{(x,y) \in \mathbb{R}^2 ; y = q\}\right)$$

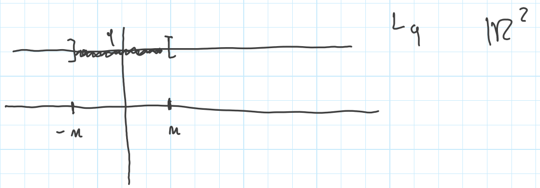


FIX ANY $m \in \mathbb{Z}^+$ AND LET

$$L_{q,m} = \{(x,y) \in \mathbb{R}^2 ; y = q , -m < x < m\}$$



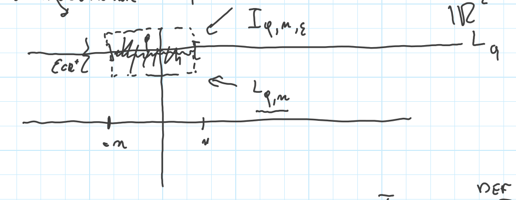
WHAT IS $\mu^*(L_{q,m})$????



NOW, GIVEN ANY $\varepsilon \in \mathbb{R}^+$, SET

$$I_{q,m,\varepsilon} = \{(x,y) \in \mathbb{R}^2 ; -m < x < m , q - \varepsilon/2 < y < q + \varepsilon/2\}$$

IN A GEOMETRIC WAY



THEN, SINCE $L_{q,n} \subseteq I_{q,n,\varepsilon} \overset{DEF}{\Longrightarrow}$

$$\mu^*(L_{q,n}) \le \mu^*(I_{q,n,\varepsilon}) = \mu(I_{q,n,\varepsilon}) \overset{DEF}{=} 2n\cdot\varepsilon = 2n\varepsilon$$

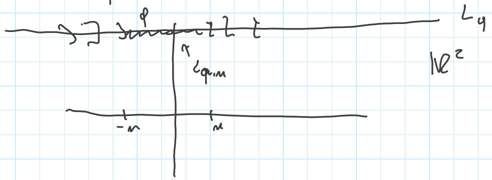
IN PLAIN WORDS $\mu^*(L_{q,n}) \le 2n\cdot\varepsilon$, $\varepsilon\in\mathbb{R}^+$ ARBITRARY

$$\Downarrow$$

$$\mu^*(L_{q,n}) = 0 \quad !!!$$

NOW FIXED $q\in\mathbb{Q}$

$$L_q = \{(x,y)\in\mathbb{R}^2 ;\ y=q\} = \bigcup_{n\in\mathbb{Z}^+}^{COUNT} L_{q,n}$$



THEN, WE APPLY THE COUNT SUBADD. PROP $\Rightarrow$

$$\underline{\mu^*(L_q)} \le \sum_{n=1}^{\infty} \underbrace{\mu^*(L_{q,n})}_{=0} = \underline{0} \quad !!!$$

APPLY AGAIN THE COUNT SUBADD PROP,

$$\mu^*(A) \overset{DEF}{=} \mu^*(\{(x,y)\in\mathbb{R}^2 ;\ y\in\mathbb{Q}\}) =$$

$$= \mu^*\Big(\bigcup_{q\in\mathbb{Q}}^{COUNT} \underbrace{\{(x,y)\in\mathbb{R}^2 ;\ y=q\}}_{=L_q}\Big)$$

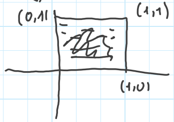
$$= \mu^*\Big(\bigcup_{q\in\mathbb{Q}}^{COUNT} \mu^*(L_q)\Big) \overset{COUNT.SUBADD.}{\le}$$

$$\le \sum_{q\in\mathbb{Q}}^{COUNT.} \underbrace{\mu^*(L_q)}_{=0} = 0 \quad !!!$$

HENCE, WE HAVE PROVED

$\mu^*(A) = 0$!!! :) !!!

$E := \{(x,y) \in \mathbb{R}^2 \,;\, 0 < x,y < 1\} \Rightarrow \mu^*(E) = 1$



$A' = A \cap E$

$\mu^*(A') = 0$!!!

STOP

QUESTIONS???

BYEBYE, GOOD WEEKEND