Wednesday, February 14, 2024 11:19 AM

RECALL : $\{A_k \subseteq \mathbb{R}^n, k \in \alpha\}$ AT MOST COUNT.

THEN

$$\mu^*\left(\bigcup_k A_k\right) \leq \sum_k \mu^*(A_k).$$

QUESTION. IF $A, B \subseteq \mathbb{R}^n$,

$$A \cap B \neq \emptyset \overset{?}{\underset{NO}{\Longrightarrow}} \mu^*(A \cup B) \overset{!}{=} \mu^*(A) + \mu^*(B) ???$$

SAY $A, B \subseteq \mathbb{R}^n$, DEFINE THEIR "DISTANCE"

$$d(A,B) \overset{\text{DEF}}{=} \inf \{ d(a,b) ; a \in A, b \in B \}$$

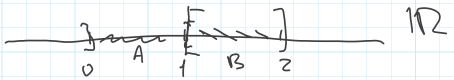
NOTICE THAT :

i) $d(A,B) > 0 \Longrightarrow A \cap B = \emptyset$.

BUT THE CONVERSE IS FALSE,

$$A \cap B = \emptyset \not\Longrightarrow d(A,B) > 0 \;!!!$$

EX. IN $\mathbb{R}$, $A =]0,1[$, $B = [1,2]$.



CLEARLY, $A \cap B = \emptyset$, BUT $d(A,B) = 0$!!!

PROP. $A, B \subseteq \mathbb{R}^n$.

IF $d(A,B) > 0$, THEN $\mu^*(A \cup B) \overset{!}{=} \mu^*(A) + \mu^*(B)$.

COROLLARY

$$\mu^*(A \cup B) < \mu^*(A) + \mu^*(B) \Rightarrow d(A,B) = 0 \;!!!$$

MEASURABLE SETS IN $\mathbb{R}^n$

AFTER CARATHEODORY, WE SAY THAT

A MEASURABLE

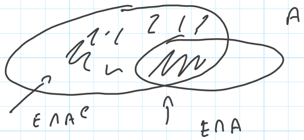
IF AND ONLY IF (DEF)

(*) $\forall E \subseteq \mathbb{R}^n$ WE HAVE

$$\mu^*(E) \overset{!!!}{=} \mu^*(E \cap A) + \mu^*(E \cap A^c) .$$

$(A^c = \mathbb{R}^n \setminus A)$

NOTICE THAT: $\forall E$



A FIXED

THEN

$$E = (E \cap A) \,\dot{\cup}\, (E \cap A^c) \xrightarrow{\text{SUBADD}}$$

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad !!!$$

NOTICE THAT, IMMEDIATELY FROM THE "SYMMETRY" OF THE AXIOM $\Longrightarrow$

A MEASURABLE $\overset{\text{THM}}{\Longleftrightarrow}$ A^c MEASURABLE.

AS AN EXAMPLE, LET US PROVE:

PROP 1. LET $A \subseteq \mathbb{R}^n$, AND ASSUME THAT $\mu^*(A) = 0$.

THEN A IS MEASURABLE.

PROOF. NOTICE THAT $\forall E \subseteq \mathbb{R}^n$

$$\mu^*(E) \leq \underset{0}{\underset{\shortparallel}{\mu^*(E \cap A)}} + \mu^*(E \cap A^c) . \qquad (**)$$

BUT $\mu^*(A) = 0$ IMPLIES THAT $\mu^*(E \cap A) = 0$, SINCE $E \cap A \subseteq A$ (MONOTONICITY)

THEN $(**)$ REDUCES TO

$$(***) \quad \mu^*(E) \leq \mu^*(E \cap A^c) .$$

BUT $E \cap A^c \subseteq E \Longrightarrow \mu^*(E \cap A^c) \leq \mu^*(E) \qquad (****)$

THEN $\mu^*(E) = \mu^*(E \cap A^c) =$

$$= \mu^*(A \cap E) + \mu^*(E \cap A^c) \qquad \text{QED}$$

COROLLARY LET $A \subseteq \mathbb{R}^n$, A COUNTABLE.

THEN $\mu^*(A) = 0$, THIS IMPLIES

THAT A MEASURABLE SET!

EX 1) $\mathbb{Q} \subseteq \mathbb{R}$, $\mathbb{Q}$ RATIONALS IS A MEAS SET.

2) $\mathbb{R} \setminus \mathbb{Q}$ IRRATIONALS IS MEASURABLE SET.

INDEED

$\mathbb{R} \setminus \mathbb{Q} = \mathbb{Q}^c$, $\mathbb{Q}$ MEASURABLE

$\Rightarrow \mathbb{R} \setminus \mathbb{Q}$ IS MEASURABLE.

3) $A = \{(x,y) \in \mathbb{R}^2 ; (x,y) \notin \mathbb{Q} \times \mathbb{Q}\}$ IS MEASURABLE.

$$A = \mathbb{R}^2 \setminus \underline{\{(x,y) \in \mathbb{R}^2 ; (x,y) \in \mathbb{Q} \times \mathbb{Q}\}}$$

↑ THIS SET IS COUNTABLE (BY FIRST CANTOR THM)

IT HAS OUTER MEASURE EQUAL TO ZERO

$\Rightarrow$ IT IS MEASURABLE

THEN, $A = \{(x,y) \in \mathbb{R}^2 ; (x,y) \notin \mathbb{Q} \times \mathbb{Q}\}$

IS MEASURABLE SINCE IT IS THE COMPLEMENTARY SET OF A MEASURABLE SET!!! QED.

FIRST OF ALL (NOTATION): IF $A \subseteq \mathbb{R}^n$ MEASURABLE

WE WRITE $\underset{\substack{\uparrow \\ \text{MEASURABLE}}}{\mu}(A) \overset{\text{DEF}}{=} \mu^*(A)$.

DEFINITION

$$\mathcal{L} = \{A \subseteq \mathbb{R}^n ; A \text{ MEASURABLE}\} \leftarrow$$

THE CLASS OF LEBESGUEAN SUBSETS

$$\cap\!|$$
$$\mathbb{P}(\mathbb{R}^n)$$

WE HAVE:

THM 1) $\emptyset \in \mathcal{L}$

CLEAR, SINCE $\mu^*(\emptyset) = 0$.

2) $A \in \mathcal{L} \Rightarrow A^c \in \mathcal{L}$

2) $A \in \mathcal{L} \Rightarrow A^c \in \mathcal{L}$.

3) $\{ A_k \subseteq \mathbb{R}^n ;\ k \in \alpha \}$ AT MOST COUNTABLE.

IF $A_k \in \mathcal{L} \Longrightarrow \bigcup_{k \in \alpha} A_k \in \mathcal{L}$

BY THE DE MORGAN LAW , 2) & 3)

4) $\{ A_k \in \mathcal{L} ;\ k \in \alpha \}$ AT MOST COUNTABLE

THEN $\bigcap_{k \in \alpha} A_k \in \mathcal{L}$ (IS MEASURABLE)

σ- ALGEBRAS IN $\mathbb{R}^n$

$\mathcal{J} \subseteq \mathbb{P}(\mathbb{R}^n)$.

$\mathcal{J}$ IS SAID TO BE A σ- ALGEBRA

WHENEVER. ???

i) $\emptyset \in \mathcal{J}$

ii) $A \in \mathcal{J} \Rightarrow A^c \in \mathcal{J}$

iii) $\{ A_k \in \mathcal{J} ;\ k \in \alpha \}$ AT MOST COUNTABLE

THEN $\bigcup_{k \in \alpha} A_k \in \mathcal{J}$.

THM 1 RESTATES

$\mathcal{L} = \{ A \subseteq \mathbb{R}^n ;\ A \text{ MEASURABLE} \}$ IS A σ- ALGEBRA !!!

$\mathcal{L} \subsetneq \mathbb{P}(\mathbb{R}^n)$, THAT IS THERE EXIST

$A \subseteq \mathbb{R}^n$, WHERE A IS NOT MEASURABLE !!!

BREAK QUESTIONS ???

BEGIN AGAIN AT 17.15.