Wednesday, February 14, 2024 11:20 AM

THM 2 $\{A_k \subseteq \mathbb{R}^n;\ A_k$ MEASURABLE$\}$ AT MOST COUNTABLE.

IF $A_i \cap A_j = \emptyset \quad \forall i \neq j$

THEN

$$\mu\left(\bigcup_k A_k\right) \overset{!}{=} \sum_k \mu(A_k).$$

EX $A = \{(x,y) \in \mathbb{R}^2;\ (x,y) \notin \mathbb{Q} \times \mathbb{Q}\}$ IS MEASURABLE.

FURTHERMORE

$$\mu(A) = +\infty.$$

INDEED $\mathbb{R}^2 = A \,\dot{\cup}\, \{(x,y) \in \mathbb{R}^2;\ (x,y) \in \mathbb{Q} \times \mathbb{Q}\} \Longrightarrow$

$$\underset{+\infty}{\underset{\parallel}{\mu(\mathbb{R}^2)}} = \mu(A) + \underbrace{\mu\left(\{(x,y) \in \mathbb{R}^2;\ (x,y) \in \mathbb{Q} \times \mathbb{Q}\}\right)}_{\underset{0}{\parallel}}$$

$$\Longrightarrow \mu(A) = +\infty$$

EX2 $\mu(\mathbb{R} \setminus \mathbb{Q}) = +\infty$.

SINCE $\underset{+\infty}{\underset{\parallel}{\mu(\mathbb{R})}} = \mu(\mathbb{R} \setminus \mathbb{Q}) + \underset{0}{\underset{\parallel}{\mu(\mathbb{Q})}}$

$$\Downarrow$$

$$\mu(\mathbb{R} \setminus \mathbb{Q}) = +\infty.$$

LIMIT THEOREMS FOR "NESTED" SEQUENCES OF MEASURABLE SETS

PROP 1 $(A_n)_{n \in \mathbb{N}}$, $A_n \subseteq \mathbb{R}^n$ MEASURABLE.

ASSUME $(A_n)_{n \in \mathbb{N}}$ "NESTED" NONDECREASING, THAT IS

$$A_n \subseteq A_{n+1} \quad \forall n \in \mathbb{N}.$$

THEN

$$\mu\left(\bigcup_{n=0}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n).$$

PROOF. CONSIDER

$(A_n)_{n \in \mathbb{N}}$ MEAS, $A_n \subseteq A_{n+1}$.

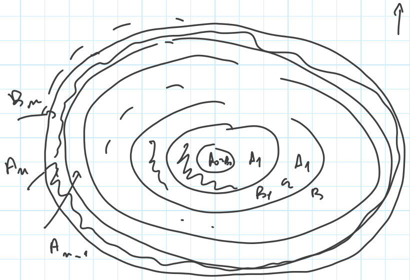
CONSIDER THE SEQUENCE

$(B_n)_{n \in \mathbb{N}}$ WHERE:

i) $B_0 = A_0$

ii) $B_n = A_n \setminus A_{n-1}$

MEAS MEAS



B_n <u>MEAS $\forall n \in \mathbb{N}$</u> !!!

<u>BUT</u> $B_i \cap B_j = \emptyset \quad \forall i \neq j$

AND $A_n = \bigcup_{i=0}^{n} B_i$.

NOW PAIRWISE DISJOINT

$$\mu\left(\bigcup_{n=0}^{\infty} A_n\right) = \mu\left(\bigcup_{n=0}^{\infty} B_n\right) \overset{\text{THM}}{=}$$

$$\overset{\text{THM}}{=} \sum_{n=0}^{\infty} \mu(B_n) \overset{\text{DEF}}{=}$$

$$\overset{\text{DEF}}{=} \lim_{n\to\infty} \left(\sum_{k=0}^{n} \mu(B_k) \right) = \lim_{n\to\infty} \mu(A_n) \quad \underline{\underline{\text{QED}}}$$

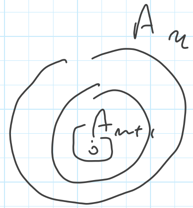
BUT, WHAT IS

$$\sum_{k=0}^{n} \mu(B_k) = \mu\left(\bigcup_{k=0}^{n} B_k\right) = \mu(A_n) \quad ??? \quad \underline{\text{OK}}$$

<u>PROP</u> LET $(A_n)_{n \in \mathbb{N}}$, $A_n \subseteq \mathbb{R}^n$ MEASURABLE.

IF $A_n \supseteq A_{n+1}$ "NESTED" NON INCREASING,

$*\quad \exists\, m \in \mathbb{N}$ s. t. $\mu(A_m) < +\infty$ (WHY ???)



THEN

$$\mu\left(\bigcap_{n=0}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

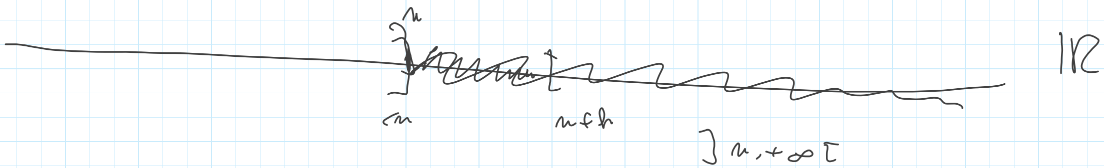
WHY ??? CONSIDER

$$A_n =]n, +\infty[\subseteq \mathbb{R} \qquad \underline{\text{MEASURABLE}}$$

BUT $\mu(A_n) = \mu(]n, +\infty[)$???

INDEED, WE HAVE

$$]n, +\infty[\supseteq]n, n+k[\qquad k \in \mathbb{Z}^+$$



THEN

$$\mu(]n, +\infty[) \geq \mu(]n, n+k[) = k \qquad \forall k \in \mathbb{Z}^+$$

$$\Rightarrow \underline{\mu(]n, +\infty[) = +\infty}.$$

BUT NOW $A_n =]n, +\infty[$ (NOTICE THAT $A_n \supseteq A_{n+1}$)

CONSIDER
∞

$$\bigcap_{n=0}^{\infty} A_n = \emptyset \quad !!!$$

$$\mu\left(\bigcap_{n=0}^{\infty} A_n\right) = \mu(\emptyset) = 0 \neq$$

$$\neq \lim_{n \to \infty} \underbrace{\mu(A_n)}_{= +\infty} = +\infty$$

CONNECTIONS BETWEEN

MEASURE THEORY AND TOPOLOGY

THM

1) ANY OPEN SET IN $\mathbb{R}^n$

IS A MEASURABLE SET !!

2) ANY CLOSED SET IN $\mathbb{R}^n$

IS A MEASURABLE SET !!!

RECALL 1) $A \subseteq \mathbb{R}^n$ $\quad$ A OPEN $\iff \forall x \in A \ \exists r \in \mathbb{R}^+$ s.t. $I(x,r) \subseteq A$.

2) B CLOSED $\overset{\text{DEF}}{\iff}$ $B^c = \mathbb{R}^n - B$ IS OPEN

RECALL

B CLOSED $\overset{\text{THEOREM}}{\iff}$ B CONTAINS ALL ITS

ACCUMULATION POINTS

WHERE

$x_0 \in \mathbb{R}^n$ IS <u>ACC.</u> <u>POINT</u> FOR $B \subseteq \mathbb{R}^n$

<u>IF AND ONLY IF</u>

$$\forall r \in \mathbb{R}^+ \quad \left(I(x_0, r) \setminus \{x_0\} \right) \cap B \neq \emptyset$$

<u>BREAK</u> QUESTIONS?

<u>BYEBYE !!!</u>