Wednesday, February 14, 2024 11:21 AM

HELLO!!!

QUESTIONS?

BEGIN?

THM 1 THE FAMILY

$$\mathcal{L} = \{ A \in \mathbb{R}^n ; \text{ A MEASURABLE} \} \subsetneq \mathbb{P}(\mathbb{R}^n)$$

IS A σ-ALGEBRA !!!

THM 2 (CONNECTION WITH "TOPOLOGY")

i) ANY OPEN SET OF $\mathbb{R}^n$ IS A MEASURABLE SET!!!

ii) ANY CLOSED SET OF $\mathbb{R}^n$ IS A MEASURABLE SET!!!

QUESTION (IN NAIVE WAY)

(σ) IF WE START FROM OPEN/CLOSED SETS AND WILL PERFORM THE "σ-ALGEBRA" OPERATION, WILL WE OBTAIN ALL THE MEASURABLE SETS IN $\mathbb{R}^n$?

($\ast\sigma$) LET US MAKE "PRECISE" QUESTION (σ) !!!

($\ast\sigma$) LEADS US TO THE NOTION OF "GENERATED σ-ALGEBRA"

THAT IS, AGAIN IN A SOMEWHAT "NAIVE" WAY:

($\sigma\sigma\sigma$) GIVEN AN ARBITRARY $\mathcal{Z} \subseteq \mathbb{P}(\mathbb{R}^n)$

THERE EXISTS THE "SMALLEST" (?) σ-ALGEBRA

$$\mathcal{S}_{\mathcal{Z}} \subseteq \mathbb{P}(\mathbb{R}^n) \qquad (\sigma\sigma\sigma)$$

S.T.

$$\mathcal{Z} \subseteq \mathcal{S}_{\mathcal{Z}} \quad ???$$

THE ANSWER TO ($\sigma\sigma\sigma$)??? IS YES !!!

THIS PROVIDED BY THE FOLLOWING

NOTION OF ($\mathcal{G} \subseteq \mathbb{P}(\mathbb{R}^n)$)

σ-ALGEBRA GENERATED BY $\mathcal{J} \subseteq \mathbb{P}(\mathbb{R}^n)$:

PRELIMINARY RMK LET

$\{\mathcal{E}_i \; ; \; i \in I\}$ AN ARBITRARY

FAMILY OF σ-ALGEBRAS IN $\mathbb{P}(\mathbb{R}^n)$.

THEN

$\mathcal{E} \overset{\text{DEF}}{=} \bigcap_{i \in I} \mathcal{E}_i$ IS STILL

A σ-ALGEBRA !!!

PROOF. WE HAVE TO PROVE:

1) $\emptyset \in \mathcal{E} \overset{\text{DEF}}{=} \bigcap_{i \in I} \mathcal{E}_i$.

BUT, WE HAVE:

$\emptyset \in \mathcal{E}_i \quad \forall i \in I \Rightarrow \emptyset \in \mathcal{E} \overset{\text{DEF}}{=} \bigcap_{i \in I} \mathcal{E}_i$. OK.

($\mathcal{E}_i$ σ-ALG)

2) WE HAVE TO PROVE:

$A \in \mathcal{E} \overset{\text{DEF}}{=} \bigcap_{i \in I} \mathcal{E}_i \overset{?}{\Rightarrow}$

$A^c \in \mathcal{E} \overset{\text{DEF}}{=} \bigcap_{i \in I} \mathcal{E}_i$. ???

BUT:

$A \in \mathcal{E} \Rightarrow A \in \mathcal{E}_i \quad \forall i \in I \xRightarrow{(\mathcal{E}_i \; \sigma\text{-ALG.})}$

$\Rightarrow A^c \in \bigcap_{i \in I} \mathcal{E}_i = \mathcal{E}$. QED.

3) LET $\{A_i \subseteq \mathbb{R}^n \; ; \; A_i \in \mathcal{E}_i, \; i \in I\}$ (†)

$\Updownarrow$

$\{A_i \subseteq \mathbb{R}^n \; ; \; A_i \in \mathcal{E} \overset{\text{DEF}}{=} \bigcap \mathcal{E}_i\}$ (††) $\overset{???}{\Rightarrow}$

(†) IMPLIES THAT

$$A_i \in \mathcal{E}_i \ \forall i \in I' \quad (††)$$

$$\Updownarrow$$

$$A_i^c \in \mathcal{E}_i \ \forall i \in I \quad (†)$$

$$\Updownarrow \qquad A_i \ \sigma\text{-ALG}$$

$$(†) \Leftrightarrow A_i^c \in \mathcal{E} \overset{DEF}{=} \bigcap_{i \in I} \mathcal{E}_i \quad \forall i \in I \ !!!$$

BUT

QED.

GIVEN $\mathcal{J} \subseteq \mathbb{P}(\mathbb{R}^n)$, LET

$\mathcal{J}_{\mathcal{J}}$ BE

$$\mathcal{J}_{\mathcal{J}} \overset{DEF}{=} \bigcap_{\mathcal{E}_i \supseteq \mathcal{J}} \mathcal{E}_i \quad \sigma\text{-ALGEBRAS} \leftarrow \text{MINIMUM ???}$$

THEN,

i) $\mathcal{J}_{\mathcal{J}}$ IS σ-ALGEBRA

AND FURTHERMORE,

ii) $\mathcal{J}_{\mathcal{J}}$ (THE σ-ALGEBRA GENERATED

$\mathcal{S}$ BY THE ARBITRARY FAMILY $\mathcal{G} \in \mathbb{P}(\mathbb{R}^n)$)

IS INDEED AND UNIQUELY DEFINED

σ-ALGEBRA SUCH σ-ALGEBRA THAT

$\mathcal{S}_{\mathcal{G}} \supseteq \mathcal{G}$ AND MINIMUM !!!

IS IT CLEAR ???

WE INTRODUCE :

LET $\mathcal{O} = \{ A \in \mathbb{R}^n ; A \text{ OPEN} \}$.

LET US CONSIDER :

$$\mathcal{B}(\mathbb{R}^n) = \mathcal{S}_{\mathcal{O}} \overset{DEF}{=} \bigcap_{\mathcal{E}_i \supseteq \mathcal{O}} \mathcal{E}_i \leftarrow \sigma\text{-ALGEBRAS}$$

BOREL σ-ALGEBRA

CLEARLY,

$$\mathcal{B}(\mathbb{R}^n) \overset{THM}{=} \bigcap \mathcal{E}_i \leftarrow \sigma\text{-ALGEBRA}$$

$\mathcal{B}(\mathbb{R}^n) \supseteq \mathcal{C}$ <u>CLOSED</u>

$\underline{\mathcal{C} = \{ C \subseteq \mathbb{R}^n ; \ C \text{ CLOSED} \}} \ldots$

<u>MAIN FACTS</u>

1) CLEARLY, $\mathcal{B}(\mathbb{R}^n) \overset{\text{TRIVIAL}}{\subseteq} \mathcal{L} =$

$\{ A \subseteq \mathbb{R}^n ; \ A \text{ MEASURABLE} \}$

BUT

ii) $\mathcal{B}(\mathbb{R}^n) \underset{!!!}{\subsetneq} \mathcal{L}$, THAT IS

$\exists \ A \subseteq \mathbb{R}^n$ S.T.:

A MEASURABLE <u>BUT</u> $A \notin \mathcal{B}(\mathbb{R}^n)$.

WE SAY THAT

$A \subseteq \mathbb{R}^n$ IS "BOREL"

$\Updownarrow$ DEF

$A \in \mathcal{B}(\mathbb{R}^n)$ ..

IN SYNTHESIS :

$\mathcal{B}(\mathbb{R}^n) \subsetneq \mathcal{L} \subsetneq \mathcal{P}(\mathbb{R}^n)$.

EX (APPL)

$A = \{ (x,y) \in \mathbb{R}^2 \ ; \ y \in \mathbb{R} \setminus \mathbb{Q} \}$.

QUESTION i) IS A MEASURABLE?

ii) IS A BOREL?

BUT, WE CAN GIVE AN ASWER TO ii) ($\Rightarrow$ TO i)) YES.

$A = \mathbb{R}^2 \setminus A^c$, WHERE

$$A^c = \{ (x,y) \in \mathbb{R}^2 \ ; \ y \in \mathbb{Q} \}$$

BUT

$$A^c = \underbrace{\bigcup_{q \in \mathbb{Q}}}^{\text{COUNT}} \{ (x,y) \in \mathbb{R}^2 \ ; \ y = q \} \quad \text{BOREL}$$

?↑ CLOSED (ELEMENTARY)

$\Downarrow$

BOREL (TRIVIAL)

$A = A^c$ ↑ A^c BOREL

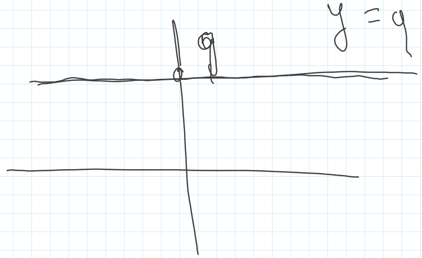
$\Downarrow$

$A = \{ (x,y) \in \mathbb{R}^2 \ ; \ y \in \mathbb{R} \setminus \mathbb{Q} \}$ IS BOREL !!!

BREAK QUESTION

IN PLAIN WORDS:

$$A^c = \{ (x,y) \in \mathbb{R}^2 ; \ \underline{y = q \in \mathbb{Q}} \}$$ IS



THE SET OF POINTS

$A^c \subseteq \mathbb{R}^2$ S.T.

GIVEN

$F : \mathbb{R}^2 \to \mathbb{R}$,

WHERE

$F(x,y) = \underline{y - q} = 0$!!!

POLYNOMIAL

$\Downarrow$

CONTINUOS

$$A^c = \{ (x,y) \in \mathbb{R}^2 ; \ F(x,y) = 0 \}$$

IS CLOSED SET

THIS IS TRUE FOR

ANY $F : \mathbb{R}^n \to \mathbb{R}$

S.T.

!!!

F CONTINUOUS :