RMKS ON GENERATED σ-ALGEBRAS

LET $\mathcal{Z} \subseteq \mathbb{P}(\mathbb{R}^n)$. THE σ-ALGEBRA GENERATED BY $\mathcal{Z}$ IS

$$\mathcal{S}_{\mathcal{Z}} \overset{\text{DEF}}{=} \bigcap_{\mathcal{Z} \subseteq \mathcal{E}} \mathcal{E}$$

$\leftarrow$ σ-ALGEBRAS $\Leftarrow$ σ-ALGEBRA

RMK 1 (COMPARISON PRINCIPLE)

$\mathcal{J}, \mathcal{Z} \subseteq \mathbb{P}(\mathbb{R}^n)$.

IF $\mathcal{J} \subseteq \mathcal{S}_{\mathcal{Z}} \Rightarrow \mathcal{S}_{\mathcal{J}} \subseteq \mathcal{S}_{\mathcal{Z}}$.

PROOF SINCE $\mathcal{J} \subseteq \mathcal{S}_{\mathcal{Z}}$ (*) $\Rightarrow$

$$\mathcal{S}_{\mathcal{J}} \overset{\text{DEF}}{=} \bigcap_{\mathcal{J} \subseteq \mathcal{E}} \mathcal{E}$$

$\leftarrow$ σ-ALG $\mathcal{S}_{\mathcal{Z}}$ APPEARS IN THIS INTERSECTIO

$$\Rightarrow \mathcal{S}_{\mathcal{J}} \subseteq \mathcal{S}_{\mathcal{Z}}$$

RMK 2 (IDENTITY PRINCIPLE) LET $\mathcal{J}, \mathcal{Z} \in \mathbb{P}(\mathbb{R}^n)$

IF

i) $\mathcal{J} \subseteq \mathcal{S}_{\mathcal{Z}}$ ($\Rightarrow \mathcal{S}_{\mathcal{J}} \subseteq \mathcal{S}_{\mathcal{Z}}$)

ii) $\mathcal{Z} \subseteq \mathcal{S}_{\mathcal{J}}$ ($\Rightarrow \mathcal{S}_{\mathcal{Z}} \subseteq \mathcal{S}_{\mathcal{J}}$)

THEN, $\mathcal{S}_{\mathcal{Z}} \overset{\text{THM}}{=} \mathcal{S}_{\mathcal{J}}$!!!

APPL

LET $\mathcal{O} = \{A \subseteq \mathbb{R}^n ; A \text{ OPEN}\}$

AND $\mathcal{C} = \{B \subseteq \mathbb{R}^n ; B \text{ CLOSE}\}$

BUT NOW, CLEARLY

(*) $\mathcal{C} \subseteq \mathcal{S}_{\mathcal{O}}$

$$\Big(\mathcal{O} \overset{\&}{\subseteq} \mathcal{S}_{\mathcal{C}}$$

$$(*) \xrightarrow{\text{IDENTITY PRINCIPLE}} \mathcal{S}_{\mathcal{O}} = \mathcal{S}_{\mathcal{C}} \overset{\text{DEF}}{=} \mathcal{B}(\mathbb{R}^n)$$

BOREL σ-ALG.

RECALL :

$$\mathcal{B}(\mathbb{R}^n) \subsetneq \mathcal{L}(\mathbb{R}^n) \subsetneq \mathbb{P}(\mathbb{R}^n)$$

NOW , CONSIDER $\mathcal{B}(\mathbb{R})$ $(n=1)$

LINDELOF LEMMA (SPECIAL FORM)

ANY OPEN SET $A \subseteq \mathbb{R}$ CAN BE EXPRESSED BY AT MUST COUNTABLE UNION OF LIMITED OPEN INTERVALS

CONSEQUENCES FIRST , CONSIDER

$$I = \{]a, b[\subseteq \mathbb{R} ;\ a, b \in \mathbb{R},\ a < b \} \subseteq \mathbb{P}(\mathbb{R}).$$

LET NOW

$\mathcal{S}_I$ THE σ-ALGEBRA GENERATED BY THE SET I !

NOW, LINDELÖF LEMMA ,

$$\mathcal{O} \subseteq \mathcal{S}_I \qquad (1)$$

BUT , TRIVIALLY ,

$$I \subseteq \mathcal{S}_{\mathcal{O}} \qquad (2)$$

BY THE IDENTITY PRINCIPLE

$$\mathcal{S}_I = \mathcal{S}_{\mathcal{O}} = \mathcal{S}_{\mathcal{C}} \overset{\text{DEF}}{=} \mathcal{B}(\mathbb{R}) \ !!!$$

IS IT CLEAR

CONTINUOUS FUNCTION $F : A \subseteq \mathbb{R}^n \to \mathbb{R}$;

OPEN SETS , CLOSED SETS

DEF $F : A \subseteq \mathbb{R}^n \to \mathbb{R}$ IS

SAID TO CONTINUOUS AT $\underline{x} \in A$

IF AND ONLY IF

$\forall \varepsilon \in \mathbb{R}^+ \quad \exists \delta \in \mathbb{R}^+$ SUCH THAT

$$|F(x) - F(\underline{x})| < \varepsilon$$

$\forall x \in I(\underline{x}, \delta) \cap A$.

LOCAL DEFINITION

WE HAVE THE GLOBAL COND.:

F CONTINUOUS ON A

IF AND ONLY IF

F IS CONTINUOUS AT $\underline{x} \in A$, $\forall \underline{x} \in A$.

THM $F : A \subseteq \mathbb{R}^n \to \mathbb{R}$.

THE FOLLOWING CONDITIONS ARE EQUIVALENT:

1) $F : A \subseteq \mathbb{R}^n \to \mathbb{R}$ IS CONTINUOUS ON A

2) $\forall B \subseteq \mathbb{R}$, B OPEN $\exists B_1 \subseteq \mathbb{R}^n$, B_1 OPEN

AND $F^{-1}[B] = A \cap B_1$

OPEN

WHERE $F^{-1}[B] \overset{DEF}{=} \{ x \in A ; F(x) \in B \}$

PREIMAGE
"FIBER"

3) $\forall C \subseteq \mathbb{R}$, C CLOSED $\exists C_1 \subseteq \mathbb{R}^n$, C_1 CLOSED

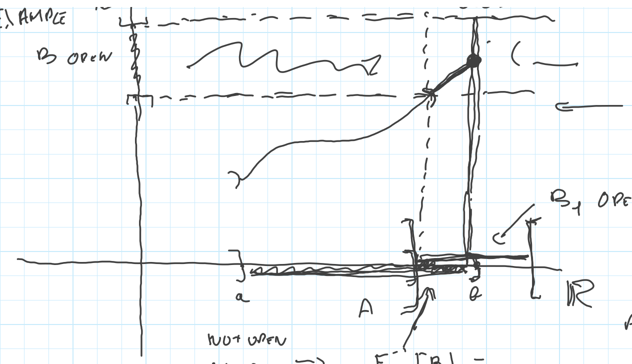
AND

$F^{-1}[C] = A \cap C_1$

$\mathbb{R}$ CLOSED

EXAMPLE

$n=1$

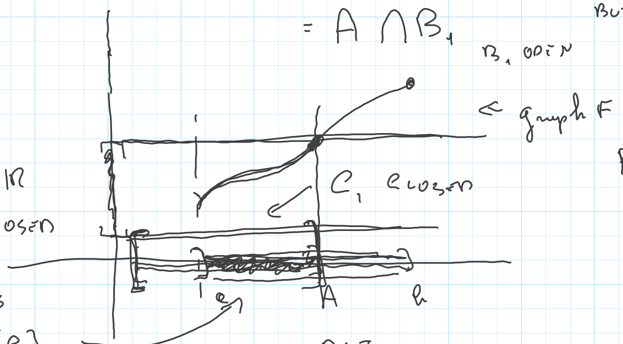


F cont. on A

NOTICE THAT A IS NEITHER OPEN BUT NOR CLOSED

NOT OPEN NOT CLOSE → $F^{-1}[B] =$

$= A \cap B_1$, B_1 OPEN



$C \subseteq \mathbb{R}$

C CLOSED

$F : A \subseteq \mathbb{R} \to \mathbb{R}$

CONT IN A

WHAT IS $F^{-1}[C]$

BUT

$F^{-1}[C] = A \cap C_1$, $C_1 \subseteq \mathbb{R}$ CLOSED

BREAK QUESTIONS?

BEGIN AGAIN AT 10.15