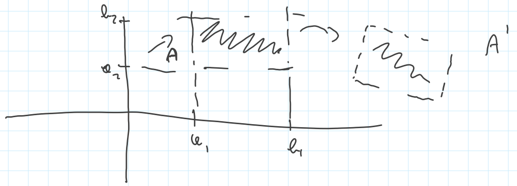
Wednesday, February 14, 2024 11:24 AM

PROBLEM

LET US CONSIDER THE SET IN $\mathbb{R}^2$



THEN $\mu(A) = (b_1 - a_1)(b_2 - a_2)$ TRIVIAL

WHAT IS $\mu(A') = ???$

INVARIANCE OF THE MEASURE !!!

GIVEN $\mathbb{R}^n$, $n \in \mathbb{Z}^1$, LET US CONSIDER A TRANSFORMATION

$T : \mathbb{R}^n \to \mathbb{R}^n$ OF AFFINE TYPE

THAT IS

$x \in \mathbb{R}^n$, $T(x) = L(x) + \underline{c}$ $\quad$ $\underline{c}$ GIVEN VECTOR $\underline{c} \in \mathbb{R}^n$

L IS A LINEAR INVERTIBLE TRANS. $L : \mathbb{R}^n \xrightarrow[1-1]{\text{on}} \mathbb{R}^n$

$\Updownarrow$ L INVERTIBLE

LINEAR MEANS:

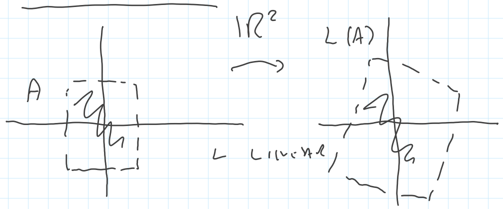
i) $L(x_1 + x_2) = L(x_1) + L(x_2)$
ADDITIVITY

ii) $L(\lambda \cdot x) = \lambda \cdot L(x)$ $\quad \lambda \in \mathbb{R}$
HOMOGENEITY

LINEAR + INVERTIBLE

$x \in \mathbb{R}^n$ $\quad T(x) = L(x) + \underline{c}$ $\leftarrow$ TRANSLATION BY $\underline{c} \in \mathbb{R}^n$. ???

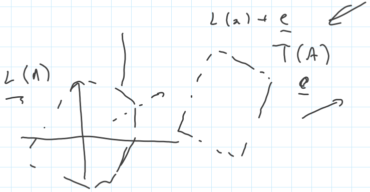
EXAMPLE IN $\mathbb{R}^2$



AFFINE TRANS

$+ \underline{c}$

TRANSLATION



NOTICE THAT :

$L : \mathbb{R}^n \xrightarrow[\text{on}]{1-1} \mathbb{R}^n \iff L$ INVERTIBLE $\iff$

LINEAR

BUT, FROM EL. LIN. ALGEBRA,

$L : \mathbb{R}^n \to \mathbb{R}^n$ LINEAR $\iff$ L IS REPRESENTED BY A $n \times n$ - MATRIX M_L WITH RESPECT TO THE CHOICE OF THE CANONICAL BASIS IN BOTH DOMAIN AND CODOMAIN $\mathbb{R}^n$

INVERTIBLE

$L : \mathbb{R}^n \to \mathbb{R}^n$ LINEAR $\iff$ L INVERTIBLE

$\Updownarrow$ LIN. ALG

$\longrightarrow \det(M_L) \neq 0$

(M_L IS NON SINGULAR).

NOW: MAIN QUESTION:

GIVEN $A \subseteq \mathbb{R}^n$, A MEASURABLE.

WHAT RELATION BETWEEN $\mu(A)$ & $\mu(T[A])$???

T AFFINE

$T[A] = \{ T(x) \; ; \; x \in A \}$. [illegible]

THM 1

$$\mu(T[A]) = |\det M_L| \cdot \mu(A)$$

$\longmapsto$

SPECIAL CASES

LET $T : \mathbb{R}^n \to \mathbb{R}$ AFFINE INVERTIBLE TRANSF

ISOMETRY

IS CALLED AN <u>ISOMETRY</u>

$$\Updownarrow$$

$$\left| \det (M_2) \right| = 1$$

$$\Updownarrow$$ LIN. ALG / GEOMETRY

$\forall x, y \in \mathbb{R}^n$

$$d(x,y) = d\big(T(x), T(y)\big) \Leftarrow$$

<u>COROLLARY OF THM 1</u> IF

$T : \mathbb{R}^n \to \mathbb{R}^n$ IS AN <u>ISOMETRY</u>

$$\Downarrow$$

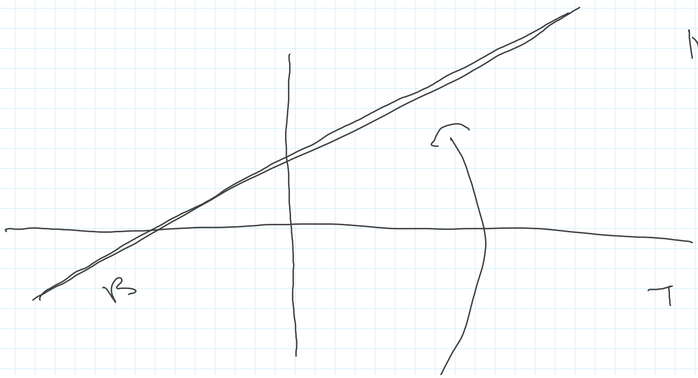
$\forall A \subseteq \mathbb{R}^n$, A MEASURABLE

$$\mu\big(T[A]\big) \overset{!}{=} \mu(A).$$

$\boxtimes$

<u>EX.</u> <u>CONSIDER</u>

$$B = \{ (x,y) \in \mathbb{R}^2 ;\ ax + by = c \}$$ $a, b \neq 0$



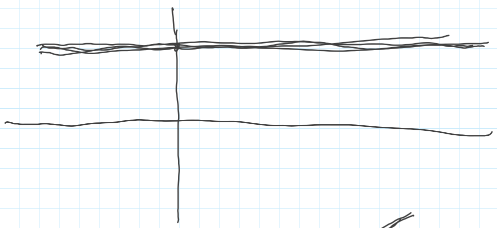
$\mathbb{R}^2$

WHAT IS $\mu(B)$???

$T : \mathbb{R}^2 \to \mathbb{R}$ <u>AFFINE TRANSF</u>

SUCH THAT

IT IS CLEAR



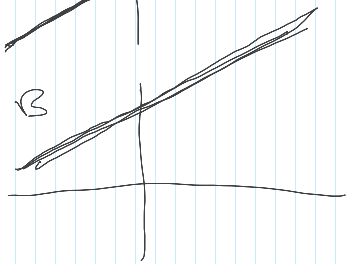
$A \in \mathbb{R}^2$

$A = \{ (x,y) \in \mathbb{R}^2 \ ; \ y = \alpha \}$

LINEAR (SUCH THAT $|\det(M_L)| = 1$)

+ C TRANSLATION

B



THM 1 $\Rightarrow$

$$\mu(B) = |\det M_L| \cdot \mu(A)$$

BUT $\mu(A) = 0$!!!

$$\Rightarrow \mu(B) = 0 \ !!! \qquad \Leftarrow$$

WHAT ARE ISOMETRIES : THAT IS

T AFFINE TRANSF SUCH THAT

$$|\det(M_L)| = 1$$

$$\Updownarrow$$

$\forall \, x, y \in \mathbb{R}^n$

$$d(T(x), T(y)) = d(x, y)$$ PRESERVE DISTANCES

RECALL PROPER ISOMETRIES

1) IF $\det(M_L) = 1 \Longrightarrow$

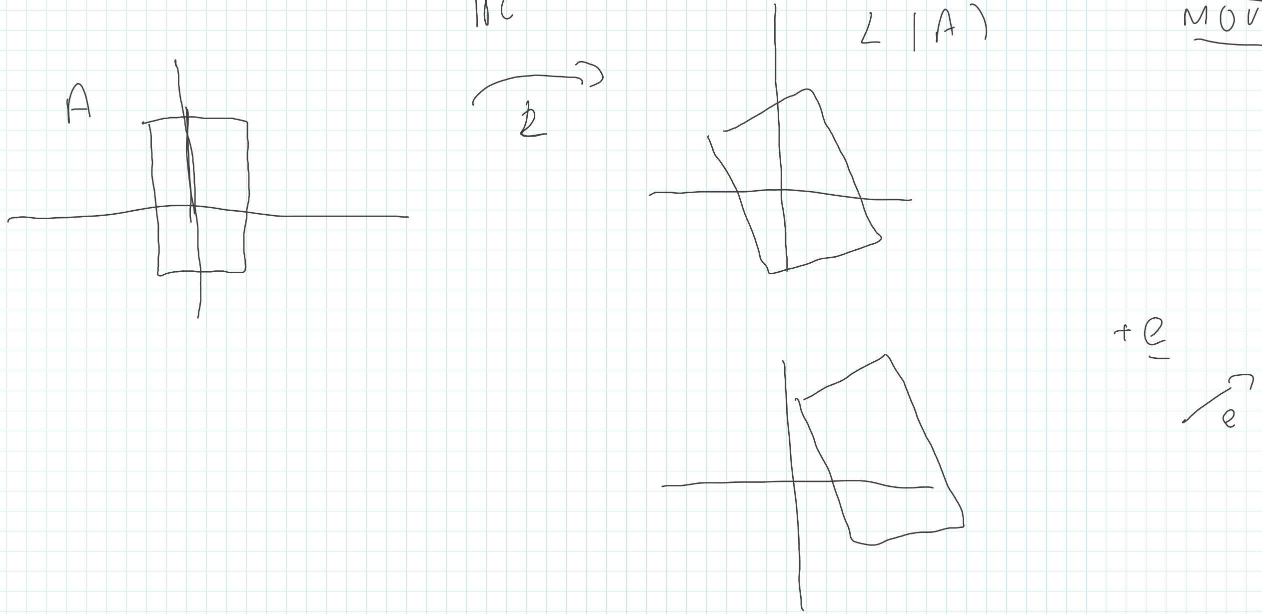
T IS A ROTOTRANSLATION

$\underset{\text{LINEAR}}{L} : \mathbb{R}^n \to \mathbb{R}^n$ IS A "ROTATION"

ALSO CALLED "EUCLIDEAN MOVEMENTS"

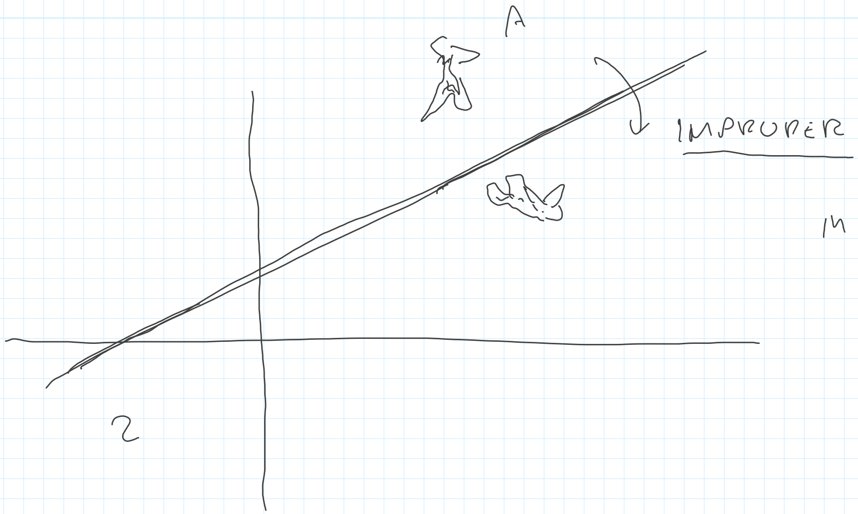
SAY

$\mathbb{R}^2$



BUT "EUCL. MOVEMENTS" $\Longleftrightarrow$ ROTOTRANSLATIONS

ARE NOT ALL THE ISOMETRIES !!!



$\mathbb{R}^2$

MAKE THE REFLECTION WITH RT. r

IT MEANS THAT ISOMETRIES CONTAIN

BOTH

i) $T: \mathbb{R}^n \to \mathbb{R}^n$ S.T $\det(M_2) = 1$ (PROPER)

BUT ALSO

ii) $T: \mathbb{R}^n \to \mathbb{R}^n$ S.T. $\det(M_1) = -1$ (IMPROPER)

WE MAKE A BREAK

QUESTIONS ?

BEGIN AT 17.20