Wednesday, February 14, 2024 11:25 AM

MEASURABLE FUNCTIONS

LET $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A MEASURABLE SET.

PROP THE FOLLOWING ARE EQUIVALENT:

i) $\forall \alpha \in \mathbb{R}$, $\{x \in A ; f(x) > \alpha\}$ IS MEASURABLE

ii) $\forall \alpha \in \mathbb{R}$, $\{x \in A ; f(x) \geq \alpha\}$ IS MEASURABLE

iii) $\forall \alpha \in \mathbb{R}$, $\{x \in A ; f(x) < \alpha\}$ IS MEASURABLE

iv) $\forall \alpha \in \mathbb{R}$, $\{x \in A ; f(x) \leq \alpha\}$ IS MEASURABLE.

MAIN DEF A FUNCTION

$f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A MEASURABLE

IS A MEASURABLE FUNCTION $\overset{DEF}{\iff}$

i), ii), iii), iv) A TRUE.

TO REWRITE i):

$\{x \in A ; f(x) > \alpha\} \overset{DEF}{=} f^{-1}[\,]\alpha, +\infty[\,]$.

SIMILARLY FOR ii), iii), iv). !!!

"SMALL" **THEOREM** BUT IMPORTANT THM

$f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A MEASURABLE.

IF f CONTINUOUS ON A, THEN f MEASURABLE.

PROOF RECALL

i) f CONT ON A

$\Updownarrow$

ii) $\forall B$ OPEN IN $\mathbb{R}$, $\exists B_1 \subseteq \mathbb{R}^n$, B_1 OPEN S.T.

$$f^{-1}[B] = A \cap B_1$$

B_1 OPEN IN $\mathbb{R}^n$

iii) $\forall C$ CLOSED IN $\mathbb{R}$, $\exists C_1 \subseteq \mathbb{R}^n$, C_1 CLOSED S.T.

$$f^{-1}[C] = A \cap C_1$$

CLOSED IN $\mathbb{R}^n$.

SO, CONSIDER $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A MEASURABLE.

CONSIDER, $\forall \alpha \in \mathbb{R}$

$$\{x \in A \;;\; f(x) > \alpha\} = f^{-1}(\underbrace{]\alpha, +\infty[}_{\text{OPEN}})$$

$\Downarrow$ (IF f IS CONTINUOUS ON A)

$\exists B_1$ OPEN IN $\mathbb{R}^n$

$$f^{-1}(]\alpha, +\infty[) = A \cap B_1 \quad \text{MEASURABLE !!!}$$

BY HYP A MEAS; B_1 OPEN IN $\mathbb{R}^n$ $\Downarrow$ B_1 MEAS

THEN, f IS A MEASURABLE FUNCTION!!!

CRUCIAL: THE CONVERSE IS FALSE, THAT IS

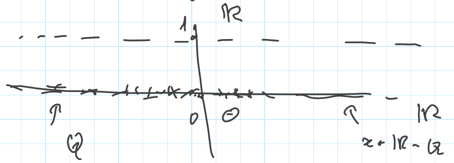
f MEASURABLE $\not\Rightarrow$ f CONTINUOUS ON A !!!

MAIN COUNTEREXAMPLE

$f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & x \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$$

THIS IS CALLED THE DIRICHLET "PATHOLOGICAL FUNCTION"



??? f IS DISCONTINUOUS EVERYWHERE

IS f DIRICHLET A MEASURABLE FUNCT?

CHECK, COND iv) THAT IS

$\forall \alpha \in \mathbb{R}$, $\{x \in A = \mathbb{R} \;;\; f(x) \leq \alpha\}$ IS MEAS ??? YES

BUT, WE CONSIDER THE FOLLOWING CASES:

1) $\alpha \geq 1$, $\{x \in \mathbb{R} ; f(x) \leq \alpha\} = \mathbb{R}$ MEASURABLE

2) $0 \leq \alpha < 1$, $\{x \in \mathbb{R} ; f(x) \leq \alpha\} = \mathbb{Q}$ MEASURABLE

3) $\alpha < 0$, $\{x \in \mathbb{R}, f(x) \le \alpha\} = \emptyset$ MEASURABLE

SOME RMKS, THAT FOLLOWS FROM OUR DISCUSSION ABOUT

$\mathcal{B}(\mathbb{R})$ BOREL σ-ALGEBRA

$\mathcal{S}_{]a,b[}$ = σ-ALGEBRA GENERATED BY LIMITED OPEN INTERVALS.

$$\mathcal{S}_{]a,b[} = \mathcal{S}_{\mathcal{O}} = \mathcal{S}_{\mathcal{C}} \overset{DEF}{=} \mathcal{B}(\mathbb{R})$$

$\mathcal{O}$ OPEN $\quad$ $\mathcal{C}$ CLOSED

(*) $\mathcal{S}_{]a,b[} \overset{THM}{=} \mathcal{B}(\mathbb{R})$

THE SAME OF

$$\mathcal{S}_{]\alpha,+\infty[} = \mathcal{S}_{[\alpha,+\infty[} = \mathcal{S}_{]-\infty,\alpha[} = \mathcal{S}_{]-\infty,\alpha]}.$$

THIS IMPLIES $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A MEASURABLE

1) f MEAS. FUNCT (BY DEF, SATISFIES i), ii), iii), iv))

$\Updownarrow$

2) $\forall B$ OPEN IN $\mathbb{R}$, $f^{-1}[B]$ MEASURABLE.

$\Updownarrow$

3) $\forall C$ CLOSED IN, $f^{-1}[C]$ MEASURABLE.

$\Updownarrow$

4) $\forall D$ BOREL SET IN $\mathbb{R}$ ($D \in \mathcal{B}(\mathbb{R})$), $f^{-1}[D]$ MEASURABLE.

CONNECTION WITH PROBABILITY THEORY AND RANDOM VARIABLE.

Ω SAMPLE SPACE $\quad \forall \alpha \in \mathbb{R}$

X IS SUCH THAT $[X < \alpha]$ IS AN EVENT

RANDOM VARIABLE

$\Updownarrow$

MEASURABLE IN THE SENS OF CARATHEODORY

NOW $X : \underbrace{\Omega}_{\text{SAMPLE SPACE}} \to \mathbb{R}$

Ω MEASURABLE

$\mu(\Omega) = 1$

$[X < \alpha] = \{x \in \Omega \,;\, X(x) < \alpha\}$ MEASURABLE

EVENT

$\Updownarrow$

$[X \leq \alpha]$ EVENT

$\Updownarrow$

$[X > \alpha]$ EVENT $\iff$ $[X \geq \alpha]$ EVENT

BREAK QUESTIONS

BEGIN AGAIN AT 17.15