Wednesday, February 14, 2024 11:26 AM

"STABILITY PROPERTIES"

OF THE CLASS OF MEASURABLE FUNCTIONS

PROP $f, g : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A MEASURABLE

f, g MEASURABLE.

i) $f+g$ MEAS

ii) fg MEAS

iii) $\lambda \in \mathbb{R}$, λf MEAS

(ALMOS EVERYWHERE) AE - TRUE

PROPERTIES ON $A \subseteq \mathbb{R}^n$

A MEASURABLE.

A PROPERTY (P) IS TRUE A.E ON A

IF AND ONLY IF

$E \subseteq A$, $E = \{ x \in A ; (P) \text{ is false on } x \}$

IS S.T. $\mu(E) = 0$

FIRST APPL.

PROP $f, g : A \to \mathbb{R}$, A MEASURABLE

SUCH THAT $f = g$ A.E.

THAT IS $E = \{ x \in A ; f(x) \neq g(x), x \in A \}$

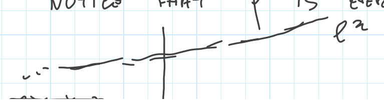
AND $\mu(E) = 0$!!!

THEN, IF f MEASURABLE, $\Rightarrow g$ MEASURABLE.

EX. LET $f : \mathbb{R} \to \mathbb{R}$ S.T.

$$f(x) = \begin{cases} e^x & x \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases} \qquad \mu(\mathbb{Q}) = 0$$

NOTICE THAT f IS EVERYWHERE DISCONTINUOUS



$x \in \mathbb{Q}$ | $\mathbb{R} \setminus \mathbb{Q}$ $\mathbb{R}$

BUT IF $g : \mathbb{R} \to \mathbb{R}$, $g(x) = e^x$

WE HAVE

1) $f = g$ A.E.

2) $g = e^x$ IS CONTINUOUS OVER $\mathbb{R} \implies$ g MEASURABLE

THEN, PROP $\implies f$ MEAS FUNCT.

RECALL : ON LIMITED + CONTINUOUS FUNCTS

$(f_n)_{n \in \mathbb{N}}$, $f_n : A \subseteq \mathbb{R}^n \to \mathbb{R}$

WE SAY THAT

f_n IS POINTWISE CONVERGENT TO $f : A \to \mathbb{R}$

IF AND ONLY IF

$$(*) \quad \forall \underline{x} \in A \left(\forall \varepsilon \in \mathbb{R}^+ \ \exists \nu_{x,\varepsilon} \in \mathbb{N} \text{ S.T. } |f(\underline{x}) - f_n(\underline{x})| < \varepsilon \quad \forall n > \nu_{x,\varepsilon} \right)$$

$\Updownarrow$ DEF

$$f_n \xrightarrow[n \to \infty]{\text{P.W.}} f \leftarrow \text{P.W. LIMIT}$$

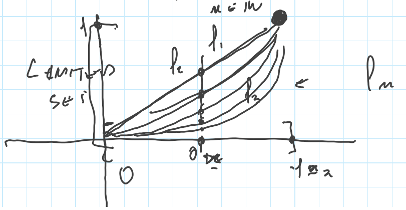
$$(**) \quad \forall \underline{x} \in A \ \left(\underset{\mathbb{R}}{f_n(\underline{x})} \longrightarrow f(\underline{x}) \right)$$

EX LET $f_n : [0,1] \to \mathbb{R}$ S.T.

$f_n(x) = x^n$

IS $(f_n)_{n \in \mathbb{N}}$ P.W. CONVERGENT? YES

WHY?



LET $x \in [0,1]$ CONSIDER TWO CASES:

i) $0 \le x < 1$

$$f_n(x) = x^n \xrightarrow[n\to\infty]{} 0$$

ii) $x = 1$

$$f_n(x) = x^n = 1 \xrightarrow[n\to\infty]{} 1$$

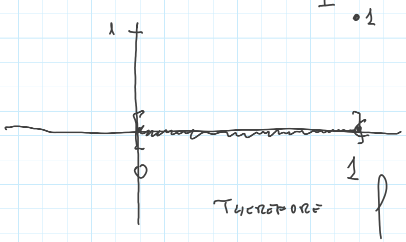
BUT NOTICE THAT:

i) $f_n \xrightarrow[n\to\infty]{PW} f$

ii) f_n LIMITED + CONTINUOUS !!!

iii) LIMIT FUNCT $f : [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$$



$\leftarrow f$

THEREFORE f IS DISCONTINUOUS AT $x = 1$

SO i) f_n LIM + CONT

ii) $f_n \xrightarrow[n\to\infty]{PW} f$

$\not\Rightarrow$

f CONTINUOUS

BREAK QUESTIONS ?

BYE BYE