Wednesday, February 14, 2024 11:28 AM

$(f_n)_{n\in\mathbb{N}}$, $f_n : A \subseteq \mathbb{R}^n \to \mathbb{R}$

DEF 1

$$f_n \xrightarrow[n\to\infty]{PW} f \overset{DEF}{\iff}$$

$$\forall x\in A \ \big(\forall \varepsilon\in\mathbb{R}^+ \ \exists\, \nu_{x,\varepsilon}\in\mathbb{N} \text{ s.t.}$$

$$|f(x)-f_n(x)| < \varepsilon \quad \forall n>\nu_{x,\varepsilon}\big)$$

$$\iff \forall x\in A \ \big(f_n(x) \xrightarrow[n\to\infty]{} f(x)\in\mathbb{R}\big)$$

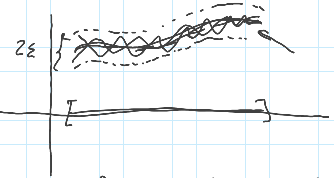
DEF 2 UNIF. CONVERGENCE

$$f_n \xrightarrow[n\to\infty]{UNIF} f \overset{DEF}{\iff} \forall \varepsilon\in\mathbb{R}^+ \ \exists \overset{!}{\nu}_\varepsilon \text{ s.t.} \qquad (*)$$

$$|f(x)-f_n(x)|<\varepsilon \quad \forall n>\nu_\varepsilon \quad \underline{\forall x\in A}$$

GEOM. INTERPRETATION

$\forall \varepsilon\in\mathbb{R}^+$



f UNIF LIMIT

$\forall n>\nu_\varepsilon$

CLEARLY: $f_n \xrightarrow[n\to\infty]{UNIF} f \implies f_n \xrightarrow[n\to\infty]{PW} f$!!!

CLEARLY AGAIN: $f_n \xrightarrow[n\to\infty]{PW} f \;\not\!\!\implies f_n \xrightarrow[n\to\infty]{UNIF} f$!!!

COUNTEREXAMPLE $f_n : [0,1]\subseteq\mathbb{R} \to \mathbb{R}$

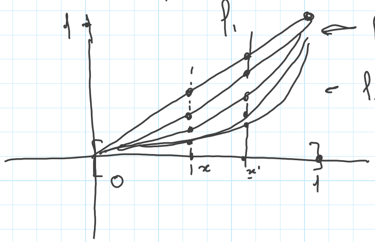
s.t. $f_n(x) = x^n \quad \forall x\in[0,1]$

f_n LIMITED + CONT.

$\big\downarrow$ PW

f

f NEED NOT TO BE CONT. !!!



GIVEN i) $x\in[0,1[$

$$f_n(x) = x^n \xrightarrow[n\to\infty]{} 0$$

ii) $x=1$ $\quad f_n(x) = 1^n = 1 \xrightarrow[n\to]{} 1$ TRIVIALLY

THEN $f:[0,1]\to\mathbb{R}$ S.T.

$$f(x)=\begin{cases}0 & 0\le x<1\\ 1 & x=1\end{cases}$$

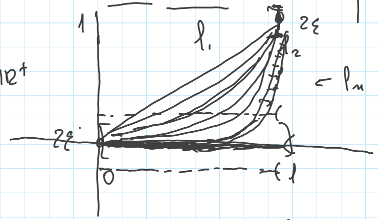
THEN f IS DISCONTINUOUS AT $x=1$.

IS OUR $(f_n)_{n\in\mathbb{Z}^+}$, $f_n:[0,1]\to\mathbb{R}$

$$f_n(x)=x^n$$

UNIF CONV TO f ???

$\varepsilon\in\mathbb{R}^+$



BUT $f(x)=\begin{cases}0 & 0\le x<1\\ 1 & x=1\end{cases}$

$$f_n\xrightarrow[n\to\infty]{PW}f \not\Rightarrow f_n\xrightarrow[n\to\infty]{UNIF}f \quad !!!$$

PROP $(f_n)_{n\in\mathbb{N}}$, $f_n:A\subseteq\mathbb{R}^n\to\mathbb{R}$ LIM + CONT

IF $f_n\xrightarrow[n\to\infty]{UNIF}f \Rightarrow f$ LIM + CONTINUOUS

MEASURABLE FUNCTIONS

FINITE CASE 1) $f_1,f_2,\dots,f_n$, $f_i:A\subseteq\mathbb{R}^n\to\mathbb{R}^n$

WE CONSIDER:

i) $\inf\{f_1,\dots,f_n\}$, $\sup\{f_1,\dots,f_n\}:A\subseteq\mathbb{R}^n\to\mathbb{R}$

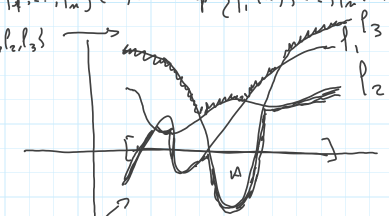
WHERE

(*) $\inf\{f_1,\dots,f_n\}(x):\overset{DEF}{=}\inf\{f_1(x),\dots,f_n(x)\}$ $\forall x\in A$

$\sup\{f_1,\dots,f_n\}(x):\overset{DEF}{=}\sup\{f_1(x),\dots,f_n(x)\}$ $\forall x\in A$

EX

$\max\{f_1,f_2,f_3\}$



$n=3$

WHAT IS $\inf\{f_1,f_2,f_3\}$

PROP IF $f_1,\dots,f_n$ ARE MEASURABLE. THEN

$\inf\{f_1,\dots,f_n\}$, $\sup\{f_1,\dots,f_n\}$ MEASURABLE.

PROP $(f_n)_{n\in\mathbb{N}}$, $f_n : A \subseteq \mathbb{R}^n \to \mathbb{R}$

SET

$\inf_n f_n$, $\sup_n f_n : A \subseteq \mathbb{R}^n \to \mathbb{R}$ S.T.

i) $\left(\inf_n f_n\right)(\underline{x}) \overset{DEF}{:=} \inf_n \left(f_n(\underline{x})\right) \quad \forall \underline{x} \in A$

ii) $\left(\sup_n f_n\right)(\underline{x}) \overset{DEF}{:=} \sup_n \left(f_n(\underline{x})\right) \quad \forall \underline{x} \in A$

IF $f_n : A \subseteq \mathbb{R}^n \to \mathbb{R}$ ARE MEASURABLE

THEN $\inf_n f_n$, $\sup_n f_n$ ARE MEASURABLE ON A.

COROLLARY $(f_n)_{n\in\mathbb{N}}$, $f_n : \underset{\uparrow \text{MEASURABLE}}{A} \subseteq \mathbb{R}^n \to \mathbb{R}$ MEASURABLE

DEFINE

$$\min_n \lim f_n = \sup_n \left(\inf_{k \geq n} f_k\right)$$

$$\max_n \lim f_n = \inf_n \left(\sup_{k \geq n} f_k\right)$$

COROLLARY BOTH

$\min\lim_n f_n$, $\max\lim_n f_n$ MEASURABLE

FURTHERMORE

$$\min_n \lim f_n \leq \max_n \lim f_n .$$

THEN $f_n \xrightarrow[n\to\infty]{PW} f \overset{THM}{\iff}$

$$\iff \min_n \lim f_n = \max_n \lim f_n \overset{DEF}{=} f \overset{\text{POP}}{=} \overset{\text{PW}}{\lim_{n\to\infty}} f_n .$$

BREAK QUESTIONS

MEAN 10.15.