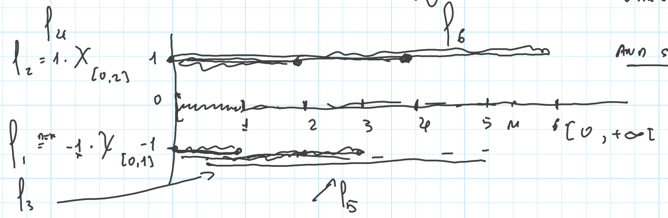
EX LET

$$f_n = (-1)^n \chi_{[0,n]} : [0,+\infty[\to \mathbb{R}, \text{ WHERE}$$

$$\chi_{[0,n]}(x) = \begin{cases} 1 & \text{IF } x \in [0,n] \\ 0 & \text{OTHERWISE} \end{cases}$$



AND SO ON.

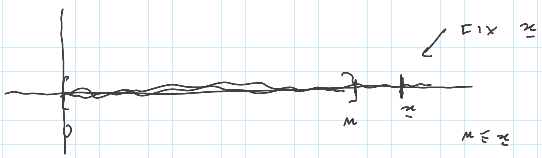
WHAT ARE

$$\min\lim_n f_n \ , \ \max\lim_n f_n \ ???$$

$$\min\lim_n f_n = \sup_n \left(\inf_{k \geq n} f_k \right) \quad \text{THAT IS}$$

$\forall\, \underline{x} \in [0,+\infty[$ $\quad \underline{x}$ FIXED.

$$\left(\min\lim_n f_n \right)(\underline{x}) = \sup_n \left(\inf_{k \geq n} f_k(\underline{x}) \right)$$



STUDY THE SEQUENCE (OF EVALUATION AT FIXED $\underline{x} \in [0,+\infty[$

$$f_n(\underline{x}) = (-1)^n \chi_{[0,n]}(\underline{x})$$

IF $n < \underline{x}$ $\quad \chi_{[0,n]}(\underline{x}) = 0$

BUT IF $n \geq \underline{x}$ THEN

$$f_n(\underline{x}) = (-1)^n \underbrace{\chi_{[0,n]}(\underline{x})}_{=1} = (-1)^n \ !!!$$

THEN, FOR FIXED $\underline{x} \in [0,+\infty[$

$$\left(f_n(\underline{x}) \right)_{n \in \mathbb{Z}^+} = \Big(\overbrace{0, 0, \underbrace{\dots, 0}_{n \leq \underline{x}}, 1, -1, 1, -1, \dots, 1 \dots}^{n=0} \Big)$$

$$\underbrace{-1, 1, -1, \dots, -1 \dots}_{n=1} \Big)$$

GIVEN ...

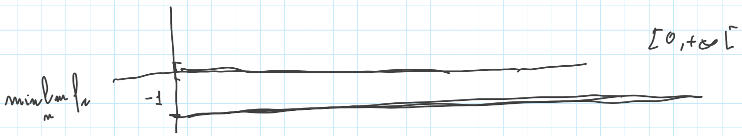
$$\inf_{k \geq n} f_k(\underline{x}) \overset{?}{=} -1 \qquad \forall n \; !!$$

$$\min\lim_n f_n(\underline{x}) = \sup_n \left(\inf_{k \geq n} f_k(\underline{x}) \right) =$$

$$= \sup_n \left(-1, -1, \dots, -1, \dots \right) = -1$$

$\Longrightarrow$ $\min\lim_n f_n$ IS COST FUNCT $\underline{-1}$

THAT IS



$$\max\lim_n f_n = \inf_n \left(\sup_{k \geq n} f_k \right) \Longrightarrow$$

$\forall \underline{x} \in [0,+\infty[$ WE HAVE $\quad$ $\underline{x}$ $\underline{\text{FIXED}}$

$$\max\lim_n f_n(\underline{x}) \overset{\text{DEF}}{=} \inf_n \left(\sup_{k \geq n} f_k(\underline{x}) \right)$$

$$f_n(\underline{x}) = (\overbrace{0, 0, \underbrace{1, 0, -1, 1, -1, 1 \dots}_{n=1}}^{n=0})$$

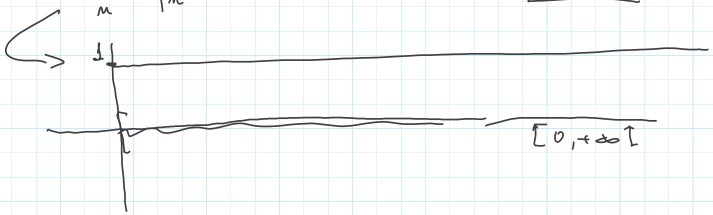
THEN $\quad \sup_{k \geq n} f_k(\underline{x}) = \underline{1} \qquad \forall n$

$$\max\lim_n f_n(\underline{x}) = \inf_n \left(\sup_{k \geq n} f_k(\underline{x}) \right) =$$

$$= \inf_n \left(1, 1, 1, \dots, 1 \right) = \underline{1} \qquad \forall \underline{x} \in [0,+\infty[$$

HENCE

$\max\lim_n f_n$ IS THE COSTANT FUNCT $\underline{= 1}$!



CLEARLY

$$\min\lim_n f_n \equiv -1 < \max\lim_n f_n \equiv 1$$

OUR SEQUENCE

$$f_n \overset{\text{DEF}}{=} (-1)^n \chi_{[0,n]}$$ IS NOT PW CONV !!!

SINCE

$$\min\lim_n f_n \underset{\neq}{<} \max\lim_n f_n .$$

LEBESGUE INTEGRALS

STEP 1 SIMPLE FUNCTS.

STEP 2 FUNCTIONS $f : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$

WHERE i) f LIMITED &

ii) $\mu(A) < +\infty$

STEP 3 $f : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ WE DROP i) & ii)

BUT ADD

f IS NONNEGATIVE ($\iff f(x) \geq 0 \ \forall x \in A$)

STEP 4 $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ IS ANY

MEASURABLE.

STEP 1 SIMPLE FUNCTION ???

CHARACTERISTIC (INDICATOR) FUNCTIONS.

SUBSET $A \subseteq \mathbb{R}^n$

$\chi_A : \mathbb{R}^n \longrightarrow \mathbb{R}$ S.T.

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

PROP

$A \subseteq \mathbb{R}^n$ IS MEASURABLE (AS A SUBSET)

IF AND ONLY IF

χ_A IS MEASURABLE (AS A FUNCTION)

PROOF $\Rightarrow$) HP A MEASURABLE

AND CONSIDER $\chi_A : \mathbb{R}^n \to \mathbb{R}$.

χ_A MEAS $\Leftrightarrow$ $\forall \alpha \in \mathbb{R}$ $\{x \in \mathbb{R}^n ; \chi_A(x) < \alpha\}$

IS MEASURABLE.

BUT CONSIDER:

i) $\alpha > 1$ $\{x \in \mathbb{R}^n ; \chi_A(x) < \alpha\} = \mathbb{R}^n$ MEAS (OK)

ii) $0 < \alpha \leq 1$, $\{x \in \mathbb{R}^n ; \chi_A(x) < \alpha\} = \mathbb{R}^n \setminus \underbrace{A}_{\text{MEAS}} = A^c$ MEAS (OK)

iii) $\alpha \leq 0$

$\{x \in \mathbb{R}^n ; \chi_A(x) < \alpha\} = \emptyset$ MEAS

DONE

$\Leftarrow$) χ_A MEAS $\Rightarrow$

$\Downarrow$

FIX $\alpha = 0$ $\{x \in \mathbb{R}^n ; \chi_A(x) > \alpha = 0\}$ MEAS

$\shortparallel$

A MEAS DONE. $\square$

BREAK QUESTIONS?

BYE BYE GOOD WEEKEND