DISCRETE MATHEMATICS

A. Brini and A. Teolis
Dipartimento di Matematica, Università di Bologna

Dedicated to the memory of Gian-Carlo Rota
Vigevano 1932 - Cambridge MA 1999

Contents

1 Introduction

The purpose of the present work is to provide short and supple teaching notes for a 30 hours introductory course on elementary *Enumerative Algebraic Combinatorics*.

We fully adopt the *Rota way* (see, e.g. [1]). The themes are organized into a suitable sequence that allows us to derive any result from the preceding ones by elementary processes.

Definitions of *combinatorial coefficients* are just by their *combinatorial meaning*. The derivation techniques of formulae/results are founded upon constructions and two general and elementary principles/methods:

- The *bad element* method (for *recursive* formulae). As the reader should recognize, the bad element method might be regarded as a combinatorial companion of the idea of *conditional probability*.

- The *overcounting* principle (for *close form* formulae).

Therefore, *no computation* is required in *proofs*: *computation formulae are byproducts of combinatorial constructions*.

We tried to provide a self-contained presentation: the only prerequisite is standard high school mathematics.

We limited ourselves to the *combinatorial point of view*: we invite the reader to draw the (obvious) *probabilistic interpretations*.

Several beautiful (and ponderous) monographs on the subject are currently available. We refer the interested reader to the ones by R. Graham, D. Knuth, O. Patashnik [2] and by Richard P. Stanley [3].

These notes are dedicated to the memory of *Gian-Carlo Rota*. Gian-Carlo was mentor of the first author and friend of both.

We quote from the obituary by *Richard P. Stanley*:

... Rota was the son of Giovanni Rota, a civil engineer and architect. Giovanni Rota was a prominent anti-fascist who had to flee Italy in 1945 *to escape Mussolini's death squads. The remarkable story of his family's escape and subsequent activities is recounted by Gian-Carlo Rota's sister Ester Rota Gasperoni in the three novels* Orage sur le Lac, L'Arbre des Capulíes, *and* L'Année américaine. *Rota ended up completing his secondary school education in Ecuador. As a result of his escape story Rota was fluent in English, Italian, Spanish, and French. In* 1950 *Rota entered* Princeton University *and graduated summa cum laude in* 1953*. He then attended graduate school at* Yale University*, receiving a* Master's Degree in Mathematics *in* 1954 *and a* Ph.D. *in* 1956 *under the supervision of Jacob T. Schwartz. After graduating from Yale, Rota received a Postdoctoral Reseach Fellowship from the* Courant Institute *at New York University. The next academic year Rota became a* Benjamin Peirce Instructor *at* Harvard University *and in* 1959 *accepted a position at the* Massachusetts Institute of Technology*. Except for a two year hiatus* $1965-67$ *at* Rockefeller University*, Rota remained at* M.I.T. *for the rest of his career. His honors and achievements include the* Colloquium Lectures of the American Mathematical Society *(*1998*), election to the* National Academy of Sciences *(*1982*), the* Leroy

P. Steele Prize for Seminal Contribution to Research *(1988),* Vice-President *of the* American Mathematical Society *(*$1995-1997$*), four honorary degrees, and the supervision of 42 Ph.D. students. He held numerous consulting positions, including a fruitful association with* Los Alamos Scientific Laboratory *officially beginning in* 1966*. He died unexpectedly in his sleep at his home in Cambridge on April* 18, 1999*.*

Rota was originally trained in functional analysis, and his early work was in this area. In the early 1960*'s he became interested in combinatorics, then a rather seedy and disreputable backwater of mathematics.*

Combinatorics is concerned with the arrangement of discrete objects and looks at such problems as the existence of an arrangement, the number or approximate number of arrangements, relations among the different arrangements, and the "optimal" arrangment according to given criteria. In general the definitions involved are easy to understand, and the arrangements have little (obvious) internal structure (Think of a jigsaw puzzle). For this reason combinatorics was not regarded by most mathematicians as a serious subject.

Rota had the vision to realize that on the contrary combinatorics had tremendous potential for elucidating and extending other areas of mathematics. He was able to recognize intuitively many problems to which combinatorics could be unexpectedly applied. As a consequence, he was the founder of the movement that lifted the subject of combinatorics to its current position as a major branch of mathematics ....

We thank Francesco (Franco) Regonati and Camilla Cobror.

We thank the former students Martin D'Ippolito and Gregorio Vettori who provided us their class notes from the course of the academic year 2020.

2 Functions between finite sets

2.1 Three elementary problems

Problem 1. Compute the number of arbitrary functions:

$$\#\{F : \underline{k} \to \underline{n}\}.$$

Problem 2. Compute the number of injective functions:

$$\#\{F : \underline{k} \stackrel{1-1}{\to} \underline{n}\}.$$

Problem 3. Compute the number of surjective functions:

$$\#\{F : \underline{k} \stackrel{su}{\to} \underline{n}\}.$$

2.2 The occupancy model

The elements of the domain set $\underline{k} = \{1, 2, \ldots, k\}$ are thought as (*labelled*) balls and the elements of the codomain set $\underline{n} = \{1, 2, \ldots, n\}$ are thought as (*labelled*) boxes.

Any function $F : \underline{k} \to \underline{n}$ gives rise to a unique distribution of the k balls into the n boxes and viceversa.

(To wit: if $i \in \underline{k}$ and $F(i) = j \in \underline{n}$, then the ball with the label i is placed into the box with the label j.)

<u>Solution of Problem 1</u>: we have n choices for the ball 1, n choices for the ball 2, ..., n choices for the ball k.

Therefore

$$\#\{F : \underline{k} \to \underline{n}\}.$$

equals

$$n \cdot n \cdots n, \quad (k \ \ times),$$

that is the *power* n^k. $\square$

<u>Solution of Problem 2</u>: Any injective function $F : \underline{k} \stackrel{1-1}{\to} \underline{n}$ gives rise to a unique distribution of the k balls into the n boxes such that any box *can contain at most one ball* and viceversa.

Hence, we have n choices for the ball 1, $n-1$ choices for the ball 2, $n-2$ choices for the ball 3, ..., $n-k+1$ choices for the ball k. Then,

$$\#\{F : \underline{k} \stackrel{1-1}{\to} \underline{n}\}.$$

equals

$$n(n-1)(n-2)\cdots(n-k+1),$$

that is the *falling factorial*

$$(n)_k \stackrel{def}{=} n(n-1)(n-2)\cdots(n-k+1).$$

Notice that, if $k = n$ then the falling factorial $(n)_n$ becomes the traditional factorial

$$n! = n(n-1)(n-2)\cdots 1.$$

As a matter of fact (since we are speaking of finite sets!), any injective function $F : \underline{k} \stackrel{1-1}{\to} \underline{n}$ (with $k = n$) is also surjective and, then, F is a bijection of $\underline{k} = \underline{n}$ to itself, that is a *permutation*.

Remark 2.1. *What about Problem 3? It has no elementary solutions (in* close form *formula)!*

We shall see (by the end of the course and using the Moebius inversion principle*) that the solution is provided by the* close form *formula:*

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j^k. \tag{1}$$

Since formula (1) *contains alternating signs (and negative integers cannot be interpreted as cardinalities), it cannot be derived by elementary constructions.*

$\square$

2.3 The word model

Beside the occupancy model, functions between finite sets admit a second (in a sense "dual") model: the elements of the domain set $\underline{k} = \{1, 2, \ldots, k\}$ can be thought as *positions* of letters in a *word of length* k, and the elements of the codomain set $\underline{n} = \{1, 2, \ldots, n\}$ can be thought as *letters* (of the alphabet $\underline{n}$). Given any function $F : \underline{k} \to \underline{n}$, we construct the word

$$W = F(1)F(2) \cdots F(k)$$

of length k on an alphabet with n letters.

For example, let $k = 4$, $n = 3$. The function

$$F : \underline{4} \to \underline{3},$$
$$F(1) = 1,\ F(2) = 3,\ F(3) = 1,\ F(4) = 2$$

gives rise to the word

$$1312.$$

Any function $F : \underline{k} \to \underline{n}$ gives rise to a unique word of length k over n letters and viceversa.

Solution of Problem 1: we have n choices for the letter to be written in position 1, n choices for the letter to be written in position 2, ..., n choices for the letter to be written in position k. Therefore

$$\#\{F : \underline{k} \to \underline{n}\}.$$

equals

$$n \cdot n \cdots n, (k\ times)$$

that is the *power* n^k. □

Solution of Problem 2: Any injective function $F : \underline{k} \stackrel{1-1}{\to} \underline{n}$ gives rise to a unique word of length k on n letters, *with no repeated letters*, and viceversa.

Hence, we have n choices for the letter to be written in position 1, $n-1$ choices for the letter to be written in position 2, $n-2$ choices for the letter to be written in position 3 ..., $n-k+1$ choices for the letter to be written in position n. Then

$$\#\{F : \underline{k} \stackrel{1-1}{\to} \underline{n}\}.$$

equals

$$n(n-1)(n-2) \cdots (n-k+1),$$

that is the *falling factorial*

$$(n)_k = n(n-1)(n-2) \cdots (n-k+1).$$

□

2.4 An elementary probalistic application: the *birthday problem*

We teach a class with k Students, say $\underline{k} = \{1, 2, \ldots, k\}$ (born in the same year, not a leap (bisextile) year).

Compute the *probability* $\mathbf{P}(\mathbf{E})$ of the event :

$$\mathbf{E} \stackrel{def}{=} \textit{ there are at least two Students with the same birthdate.}$$

The *date of birth* is a function from the set of Students $\underline{k} = \{1, 2, \ldots, k\}$ to the set of the days of the year $\underline{365} = \{1, 2, \ldots, 365\}$:

$$F : \underline{k} = \{1, 2, \ldots, k\} \to \underline{365} = \{1, 2, \ldots, 365\},$$

and the event $\mathbf{E}$ can be formalized in the following way:

$$\mathbf{E} \ = \ \{F : \underline{k} \to \underline{365};\ F \textit{ not injective}\}.$$

Then, the complementary event is:

$$\mathbf{E}^c \ = \ \{F : \underline{k} \stackrel{1-1}{\to} \underline{365}\}$$

and, hence,

$$\mathbf{P}(\mathbf{E}) \ = \ 1 - \mathbf{P}(\mathbf{E}^c).$$

The probability of $\mathbf{P}(\mathbf{E}^c)$ equals

$$\frac{|\{F : \underline{k} \stackrel{1-1}{\to} \underline{365}\}|}{|\{F : \underline{k} \to \underline{365}\}|} \ = \ \frac{(365)_k}{365^k},$$

then

$$\mathbf{P}(\mathbf{E}) \ = \ 1 - \frac{(365)_k}{365^k}.$$

Amazingly, it follows that for $k \geq 23$ (at least 23 Students) this probability *is greater than* $\frac{1}{2}$.

3 Binomial coefficients

3.1 Subsets and characteristic functions

Le X be a finite set, $|X| = n$.

Given a subset $A \subseteq X$, the *characteristic function* of A is the function

$$\chi_A : X \to \{0, 1\}$$

such that

$$\chi_A(x) = 1 \ if \ x \in A, \quad \chi_A(x) = 0 \ if \ x \notin A.$$

Given a function

$$\chi : X \to \{0, 1\},$$

its *support* is the subset

$$supp(\chi) = \{x \in X;\ \chi(x) = 1\} \subseteq X.$$

The "construction" (as a matter of fact: function)

$$C_1 : A \mapsto \chi_A$$

and the "construction"

$$C_2 : \chi \mapsto supp(\chi)$$

are easily recognized to provide a pair of *inverse maps*:

$$C_1 : \mathbb{P}(X) \stackrel{def}{=} \{A;\ A \subseteq X\} \to \{\chi; \chi : X \to \{0, 1\}\}$$

and

$$C_2 : \{\chi;\ \chi : X \to \{0, 1\}\} \to \mathbb{P}(X) \stackrel{def}{=} \{A;\ A \subseteq X\}.$$

In details:

$$C_2(C_1(A)) = C_2(\chi_A) = supp(\chi_A) = A,$$

and

$$C_1(C_2(\chi)) = C_2(supp(\chi)) = \chi_{supp(\chi)} = \chi.$$

Hence, C_1 and C_2 are *bijections*. Then the two sets are equicardinal:

$$|\mathbb{P}(X)| = |\{\chi;\ \chi : X \to \{0, 1\}\}|.$$

The cardinality of the second set equals 2^n, by the solution to Problem 1.

Proposition 3.1. *Let* $|X| = n$. *Then*

$$|\mathbb{P}(X)| = 2^n.$$

□

3.2 Binomial coefficients: the combinatorial definition

Let $n, k \in \mathbb{N}$ be natural integers.

The *binomial coefficient*

$$\binom{n}{k}$$

is defined by means of its combinatorial meaning:

$$\binom{n}{k} \stackrel{def}{=} \#\ \text{k-subsets of an n-set}.$$

Let X be a finite set, $|X| = n$. Since $\mathbb{P}(X) \stackrel{def}{=} \{A;\ A \subseteq X\}$ equals the *disjoint union*:

$$\mathbb{P}(X) = \dot\cup_{k=0}^{n}\ \{A \subseteq X;\ |A| = k\},$$

from Proposition 3.1 we immediately have:

Corollary 3.2.

$$\sum_{n=0}^{n} \binom{n}{k} = 2^n.$$

$\square$

3.3 Dispositions with no repetitions and increasing words

Let $A = \{a_1 < a_2 < \ldots < a_n\}$ be an *alphabet* on n letters, that is a finite n-set endowed with a total order $<$.

A word of length k on A, say

$$w = a_{i_1} a_{i_2} \ldots a_{i_k}$$

is *increasing* whenever $a_{i_1} < a_{i_2} < \cdots < a_{i_k}$.

Clearly, given an increasing word of length k on n letters, its set of letters is a k-subset of the n-set $A = \{a_1 < a_2 < \ldots < a_n\}$, and, conversely, given a k-subset of the n-set $A = \{a_1 < a_2 < \ldots < a_n\}$ we can write its elements (in a unique way) in increasing order, therefore obtaining an increasing word of length k.

Then the two families are *bijectively equivalent* and, a fortiori, they have the same cardinality $\binom{n}{k}$. In the language of old fashioned Combinatorial Calculus, increasing words are called *dispositions with no repetitions.*

Example 3.3. Let $A = \{a_1 < a_2 < a_3 < a_4 < a_5\}$ and let

$$w = a_1 a_3 a_4$$

be an increasing word of length 3. It bijectively corresponds to the 3-subset $\{a_1, a_3, a_4\}$.

$\square$

3.4 The overcounting principle (*shepherd's principle*)

We introduce a general principle that will be systematically used throughout our presentation. As a fairy tale of the mathematical folklore, it is usually known as the *shepherd's principle.* To wit: *A shepherd has to count the sheep of his flock, how does he proceed? Count the number of legs then divide by four!* This metaphor seems to express a paradoxical procedure, however it is actually very profound and effective. Suppose we have to enumerate objects of a certain type (sheep, in the metaphor), but we don't know how. Suppose that each sheep has a fixed k number of other objects associated with it (the legs, in the metaphor) and these objects are easier to count. So we count the legs and then divide by k (Eureka!)

In the next paragraph we will immediately see a significant application.

3.5 On the computation of binomial coefficients: *close form* formulae

Our problem is to find algebraic formulas (*close form* formulae) to calculate the binomial coefficients.

We want to apply the overcounting/shepherd's principle. We have to ask ourselves: if the k-subsets are sheep, what are the legs?

Let $n \in \mathbb{N}$ and $\underline{n} = \{1, 2, \ldots, n\}$ the standard n-set. Given an injective function $F : \underline{k} \stackrel{1-1}{\to} \underline{n}$, consider its image

$$Im(F) \stackrel{def}{=} \{F(i);\ i \in \underline{k}\} \subseteq \underline{n}.$$

Since F is injective, then its image $Im(F)$ is a k-subset of $\underline{n}$, and futhermore any k-subset of $\underline{n}$ can be obtained as the image of a suitable injective function from $\underline{k}$ to $\underline{n}$. Well, the injective functions from $\underline{k}$ to $\underline{n}$ are the legs! Now the question is: how many legs per sheep? In precise terms, it becomes: *how many injective functions have the same image*?

It is easy to recognize that, given $F, G : \underline{k} \stackrel{1-1}{\to} \underline{n}$ we have $Im(F) = Im(G)$ if and only if there exists a permutation σ of $\underline{k}$ such that $F = G\ \sigma$.

There are exactly $k!$ *legs for each sheep*!

Therefore

$$\binom{n}{k} = \frac{|\{F : \underline{k} \stackrel{1-1}{\to} \underline{n}\}|}{k!}.$$

By the solution to Problem 2, we infer:

Proposition 3.4.

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}.$$

□

3.6 Binomial coefficients: recursive computation

Binomial coefficients are regarded as a *double sequence*:

$$\left(\binom{n}{k} \right)_{n,k \in \mathbb{N}}.$$

Then, it is convenient to represent them by means of a *biinfinite matrix*, that is a function

$$M : \mathbb{N} \times \mathbb{N} \to \mathbb{R}, \quad M : (n, k) \mapsto \binom{n}{k}.$$

In plain words, we arrange the binomial coefficients in the following way:

$$
\begin{array}{c|cccccccc}
 & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \cdots & \boldsymbol{k} & \cdots \\
\hline
\mathbf{0} & \binom{0}{0} & \binom{0}{1} & \binom{0}{2} & \binom{0}{3} & \binom{0}{4} & \cdots & \binom{0}{k} & \cdots \\
\mathbf{1} & \binom{1}{0} & \binom{1}{1} & \binom{1}{2} & \binom{1}{3} & \binom{1}{4} & \cdots & \binom{1}{k} & \cdots \\
\mathbf{2} & \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & \binom{2}{3} & \binom{2}{4} & \cdots & \binom{2}{k} & \cdots \\
\mathbf{3} & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & \binom{3}{4} & \cdots & \binom{3}{k} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\boldsymbol{n} & \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \binom{n}{4} & \cdots & \binom{n}{k} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}.
$$

The elements of the 0-row are:

$$\binom{0}{k} \overset{def}{=} \#\ \text{k-subsets of the 0-set } (\textit{the empty set } \emptyset).$$

Clearly $\binom{0}{0} = 1$ and $\binom{0}{k} = 0$ whenever $k > 0$. By using the Kronecker δ symbol, we write:

$$\binom{0}{k} = \delta_{0,k}. \tag{2}$$

The elements of the 0-column are:

$$\binom{n}{0} \overset{def}{=} \#\ \text{0-subsets of a n-set.}$$

Clearly, the unique 0-set is the empty set $\emptyset$: then,

$$\binom{n}{0} = 1 \ for\ every\ n \in \mathbb{N}. \tag{3}$$

Do binomial coefficients obey some kind of recursion? To deal with this problem, we will use the so called *"bad element" method*.

3.6.1 Linear recursions and the *bad element* method

We must count the elements of a variety V of constructions that can be given on a set of n elements. We choose an element, for example the last one, that we will call the *"bad element"*. We divide our variety into disjoint and exhaustive classes with respect to the behavior of the "bad element". Clearly, the cardinality of the variety V will be given by the sum of the cardinalities of the subclasses and counting these cardinalities will involve reasoning only on the first $n-1$ elements. Let's immediately see a first and prototypical application.

3.6.2 The Pascal/Tartaglia/Stifel/Chu recursion for binomial coefficients

To calculate $\binom{n}{k}$ we have to calculate (from the definition itself) the cardinality:

$$|\{A \subseteq \underline{n};\ |A| = k\}|.$$

In the set $\underline{n} = \{1, 2, \ldots, n\}$, choose as "bad element" the last element n (this is an arbitrary choice).

For the family

$$\{A \subseteq \underline{n};\ |A| = k\}$$

we have two (disjoint) cases:

i) $n \notin A$. In this case, A is a subset of $\underline{n-1} = \{1, 2, \ldots, n-1\}$, with $|A| = k$.

The cardinality of this class is:

$$\binom{n-1}{k},$$

by definition.

ii) $n \in A$. In this case, A can be (uniquely) expressed in the form:

$$A = \ A' \dot{\cup} \{n\},$$

with

$$A' \subseteq \underline{n-1}, \quad |A'| = k-1.$$

The cardinality of this class is:

$$\binom{n-1}{k-1}$$

by definition.

Therefore, we get the famous recursion (known to the ancient civilizations B.C. of the Far East!):

Proposition 3.5. *We have*

$$\binom{n}{k} \overset{T\underline{H}M}{=} \binom{n-1}{k-1} + \binom{n-1}{k}.$$

□

This recursion, together with the initial condition (2) and (3) allows us to compute the entries of the matrix of binomial coefficients in an effective way.

We have

	0	**1**	**2**	**3**	**4**	$\cdots$
0	**1**	**0**	**0**	**0**	**0**	$\cdots$
1	**1**	**1**	**0**	**0**	**0**	$\cdots$
2	**1**	**2**	**1**	**0**	**0**	$\cdots$
3	**1**	**3**	**3**	**1**	**0**	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

.

3.7 Graphs

A (labelled) *graph* is - roughly speaking - a finite set of vertices $V = \{1, 2, \ldots, n\}$ joined by (may be intersecting) *edges*; two vertices joined by an edge are said to be *adjacent*. The edges are identified with *nonordered* couples $\{i, j\}$, $i, j \in V = \{1, 2, \ldots, n\}$.

Therefore, we formalize the notion of a graph G in the following way.

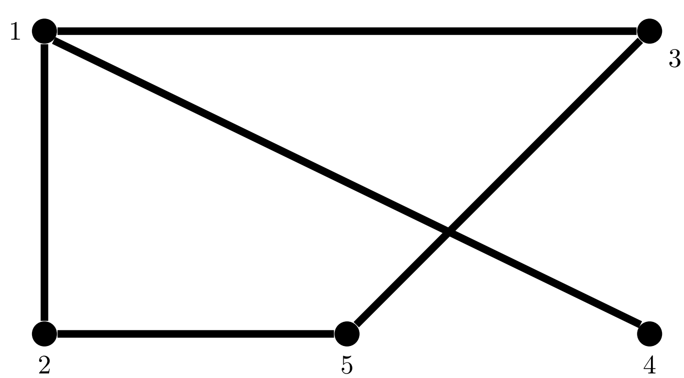
A graph G is a pair

$$G = (V, E),$$

where $V = \{1, 2, \ldots, n\}$ is the set of vertices and the set E of edges is

$$E \subseteq \{A \subseteq V;\ |A| = 2\}.$$

Example 3.6. Consider the graph G:



where

$$V = \{1,2,3,4,5\}, \quad E = \{\ \{1,2\},\{1,3\},\{1,2\},\{1,4\},\{2,5\},\{3,5\}\ \}.$$

$\square$

Proposition 3.7. *The number of graphs G on* n *vertices is*

$$2^{\binom{n}{2}}.$$

Proposition 3.8. *The number of graphs G on* n *vertices with exacly k edges is*

$$\binom{\binom{n}{2}}{k}.$$

3.8 Digraphs

A (labelled) *directed graph* (*digraph*, for short) is, roughly speaking, a finite set of *vertices* $V = \{1,2,\ldots,n\}$ joined by (may be intersecting) *arrows*. Then, an arrow ($i \to j$, with *head* i and *tail* j) is identified with the *ordered* pair (i,j), $i,j \in V = \{1,2,\ldots,n\}$.

An arrow $i \to i$, with the same *head* i and *tail* i is called a *loop*.

Therefore, we formalize the notion of a digraph $\vec{G}$ in the following way.

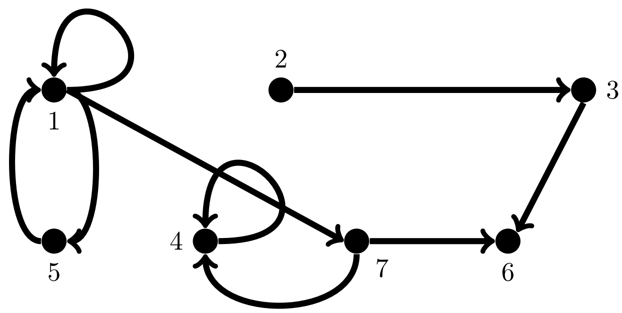
A digraph $\vec{G}$ is a pair

$$\vec{G} = (V, \vec{E}),$$

where $V = \{1,2,\ldots,n\}$ is the set of vertices and the set $\vec{E}$ of arrows is

$$\vec{E} \subseteq\ V \times V.$$

Example 3.9. Consider the digraph $\vec{G}$:



where

$$V = \{1,2,3,4,5,6,7\},$$

$$\overrightarrow{E} = \{\ (1,1),(1,5),(5,1),(1,7),(7,4),(4,4),(2,3),(3,6),(7,6)\ \}.$$

$\square$

Then,

Proposition 3.10. *The number of digraphs* $\overrightarrow{G}$ *on* n *vertices is*

$$2^{n^2}.$$

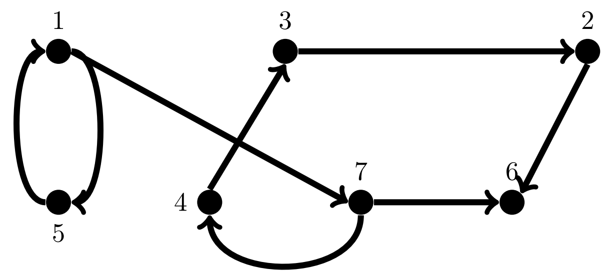
Proposition 3.11. *The number of digraphs* $\overrightarrow{G}$ *on* n *vertices with exacly* k *arrows is*

$$\binom{n^2}{k}.$$

Clearly, a digraph $\overrightarrow{G} = (V, \overrightarrow{E})$ has *no loops* whenever

$$\overrightarrow{E} \subseteq\ V \times V - \{(i,i);\ i \in V\}.$$

Example 3.12.



where

$$V = \{1, 2, 3, 4, 5, 6, 7\},$$

$$\vec{E} = \{\ (1,5), (5,1), (1,7), (7,4), (4,3), (3,2), (2,6), (7,6)\ \}.$$

$\square$

Then,

Proposition 3.13. *The number of digraphs* $\vec{G}$ *with no loops on* n *vertices is*

$$2^{n(n-1)}.$$

Proposition 3.14. *The number of digraphs* $\vec{G}$ *with no loops on* n *vertices with exactly* k *edges is*

$$\binom{n(n-1)}{k}.$$

4 Recursive matrices and generating functions

4.1 The algebra of formal power series $\mathbb{R}[[t]]$

Let

$$(a_n)_{n \in \mathbb{N}} = (a_0, a_1, \ldots, a_n, \ldots), \quad a_n \in \mathbb{R} \tag{4}$$

be a sequence with real entries.

Let t be a "formal" variable.

The associated *formal power series* is the "expression"

$$\alpha(t) = \sum_{n=0}^{\infty} a_n t^n. \tag{5}$$

The series (5) is also called the *generating series* of the sequence (4).

Note that *polynomials* are special cases of "finite" formal power series. To wit: in the case of polynomials all but a finite number of the coefficients a_n are ZERO. In a formal way:

$$\alpha(t) = \sum_{n=0}^{\infty} a_n t^n,$$

but there exists $\underline{n}$ such that

$$a_n = 0, \quad for\ every\ n > \underline{n}.$$

Formal power series can be summed:

$$\alpha(t) = \sum_{n=0}^{\infty} a_n t^n, \quad \beta(t) = \sum_{n=0}^{\infty} b_n t^n,$$

then, by definition:

$$\alpha(t) + \beta(t) = \sum_{n=0}^{\infty} (a_n + b_n) t^n.$$

Formal power series can be multiplied by a scalar factor $\lambda \in \mathbb{R}$:

$$\lambda\alpha(t) = \sum_{n=0}^{\infty} (\lambda a_n) t^n.$$

The set of formal power series $\mathbb{R}[[t]]$ endowed with these operations is clearly a vector space. Its <u>zero vector</u> is the *identically zero series*:

$$\underline{0}(t) = \sum_{n=0}^{\infty} \underline{0}_n t^n, \ \underline{0}_n = 0 \in \mathbb{R}, \ for\ every\ n \in \mathbb{N}.$$

But formal power series can be multiplied together, too. To wit:

$$\alpha(t) = \sum_{n=0}^{\infty} a_n t^n, \quad \beta(t) = \sum_{n=0}^{\infty} b_n t^n,$$

then, by definition, the *product series*

$$\alpha(t)\beta(t)$$

is the series

$$\gamma(t) = \sum_{n=0}^{\infty} c_n t^n,$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}. \tag{6}$$

Notice that this multiplication rule is nothing but the natural generalization of the ordinary one for polynomials (from high school mathematics)!

The vector space $\mathbb{R}[[t]]$ - endowed with this product operation - turns out to be an *ALGEBRA*, i.e. the above product is *associative* and *distributes* w.r.t. to vector space operations (i.e., addition and scalar multiplication).

Futhermore, $\mathbb{R}[[t]]$ is a *commutative* algebra, that is:

$$\alpha(t)\ \beta(t) = \beta(t)\ \alpha(t).$$

Notice that $\mathbb{R}[[t]]$ has a *unit*, which is the multiplicative neutral element $\underline{1}(t)$:

$$\underline{1}(t)\ \alpha(t) = \alpha(t) = \alpha(t)\ \underline{1}(t).$$

Clearly, $\underline{1}(t)$ is the *constant series*:

$$\underline{1}(t) = 1,$$

that is

$$\underline{1}(t) = \sum_{n=0}^{\infty} \underline{1}_n t^n,$$

with $\underline{1}_0 = 1$, $\underline{1}_n = 0$ for $n > 0$.

Remark 4.1. *As a vector space, the space* $\mathbb{R}[[t]]$ *of formal power series is the* dual space $(\mathbb{R}[t])^*$ *of the vector space of polynomials* $\mathbb{R}[t]$. *Notice that* $\mathbb{R}[t]$ *is not of finite dimension. As a matter of fact,* $\mathbb{R}[t]$ *has infinite* countable *dimension. Indeed, a basis of* $\mathbb{R}[t]$ *is provided by the family of* power monomials*:*

$$\{t^n; n \in \mathbb{N}\} = \{1, t, t^2, \ldots, t^n, \ldots\}$$

The dimension of $\mathbb{R}[[t]] = (\mathbb{R}[t])^*$ *is* more than countable. *We recall that, in infinite dimension, the basis theorem for vector spaces is just an* existence *theorem (the standard proof involves the* Zorn Lemma*). Indeed, an explicit basis of* $\mathbb{R}[[t]]$ *is still unknown.*

□

4.2 Row generating series (functions) and recursive matrices

Let

$$M : \mathbb{N} \times \mathbb{N} \to \mathbb{R}, \quad M : (n, k) \mapsto M(n, k) \in \mathbb{R}$$

be a biinfinite matrix. Alternatively, we denote this matrix as follows:

$$M = \left[\ M(n, k)\ \right]_{n,k \in \mathbb{N}}.$$

Given $n \in \mathbb{N}$, the nth *generating series* of the matrix M is the formal power series

$$M_n(t) = \sum_{k=0}^{\infty} M(n, k) t^k, \tag{7}$$

the generating series of the sequence in the nth row.

The matrix M is said to be a *recursive matrix* whenever the following condition holds:

$$M_n(t) = (M_1(t))^n . \tag{8}$$

Clearly, condition (8) is equivalent to the conditions:

$$M_0(t) = 1, \tag{9}$$

$$M_n(t) = M_1(t) \cdot M_{n-1}(t), \ \textit{for } n > 1. \tag{10}$$

The series $M_0(t)$ and the series $M_1(t)$ are called the *initial condition* and the *recursion rule* of the recursive matrix M, respectively.

4.2.1 The matrix of binomial coefficients as a recursive matrix

We have:

Theorem 4.2. *The matrix of binomial coefficients*

$$M = \left[\ M(n,k) \ \right]_{n,k\in\mathbb{N}} \stackrel{def}{=} \left[\binom{n}{k} \right]_{n,k\in\mathbb{N}}$$

is a recursive matrix *having as recursion rule the polynomial*

$$M_1(t) = 1 + t.$$

Proof. We verify conditions (10). Since $\binom{0}{n} = \delta_{0,n}$, then $M_0(t) = 1$.

Since $\binom{1}{0} = \binom{1}{1} = 1$ and $\binom{1}{n} = 0$ for $n > 1$, then

$$M_1(t) \stackrel{def}{=} \sum_{n=0}^{\infty} \binom{1}{n} t^n = 1 + t.$$

We have to prove that

$$M_n(t) \stackrel{def}{=} \sum_{k=0}^{\infty} \binom{n}{k} t^k$$

equals

$$M_1(t) \cdot M_{n-1}(t) \stackrel{def}{=} (1+t) \cdot \left(\sum_{k=0}^{\infty} \binom{n-1}{k} t^k \right).$$

Write

$$M_1(t) \cdot M_{n-1}(t) = \gamma(t) = \sum_{k=0}^{\infty} c_k t^k.$$

By the multiplication rule (6) of series, we have:

$$c_k \stackrel{def}{=} \binom{1}{0}\binom{n-1}{k} + \binom{1}{1}\binom{n-1}{k-1}.$$

But
$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$
by Proposition 3.5. Then
$$M_1(t) \cdot M_{n-1}(t) = \gamma(t) = \sum_{k=0}^{\infty} \binom{n}{k} t^k \stackrel{def}{=} M_n(t).$$
□

Corollary 4.3. ***(Binomial Theorem)*** *We have:*
$$(1+t)^n \;=\; \sum_{k=0}^{n} \binom{n}{k} t^k.$$

4.2.2 The generalized Vandermonde convolutions for recursive matrices

Let
$$M = \left[\ M(n,k)\ \right]_{n,k \in \mathbb{N}}.$$
be a recursive matrix.

We have:

Proposition 4.4. ***(General Vandermonde convolutions)*** *Let $i, j \in \mathbb{Z}^+$ and $n = i + j$. Then*
$$M(n,k) = \sum_{h=0}^{k} M(i,h)M(j,k-h). \tag{11}$$

Proof. Since the matrix M is a recursive matrix, then
$$M_n(t) = (M_1(t))^n = (M_1(t))^i \cdot (M_1(t))^j = M_i(t) \cdot M_j(t).$$
Write
$$M_i(t) \cdot M_j(t) \stackrel{def}{=} \gamma(t) = \sum_{k=0}^{\infty} c_k t^k.$$
By the multiplication rule (6) of series, we have:
$$c_k = \sum_{h=0}^{k} M(i,h)M(j,k-h).$$
But
$$\gamma(t) = \sum_{k=0}^{\infty} c_k t^k = M_n(t) \stackrel{def}{=} \sum_{k=0}^{\infty} M(n,k) t^k$$
and thus the assertion follows. □

In the special case

$$M = [\ M(n,k)\ ]_{n,k\in\mathbb{N}} = \left[\binom{n}{k}\right]_{n,k\in\mathbb{N}},$$

Proposition 4.4 yields

Corollary 4.5. *Let $i, j \in \mathbb{Z}^+$ and $n = i + j$. Then*

$$\binom{n}{k} = \sum_{h=0}^{k} \binom{i}{h}\binom{j}{k-h}. \tag{12}$$

5 Multisets and multiset coefficients

5.1 The problem of *rows*

We consider the following problem: *In how many ways can we arrange k objects into n different* rows*?*

We can think of the k objects as "flags" and the n rows as "flagpoles".

We fix the number n of flagpoles and try to compute the value

$$L_k \stackrel{def}{=} \#\ \text{of ways to arrange k flags},$$

as a function of k. We can do this by meaning of recursion. Clearly, $L_1 = n$ by definition.

We shall use the "bad element" method and choose the last flag (with label k) as "bad element".

Suppose the first $k-1$ flags are already placed on the n flagpoles (this can be done in L_{k-1} ways). Clearly, there will be i_1 flags on flagpole 1, i_2 flags on flagpole 2, . . . , i_n flags on flagpole n, with

$$i_1 + i_2 + \cdots + i_n = k - 1.$$

In how many ways can we insert the last flag k? Clearly, there will be $i_1 + 1$ choices on flagpole 1, i_2+1 choices on flagpole 2, . . . , i_n+1 choices on flagpole n. Therefore, the total number of ways in which we can insert the last flag k will be given by:

$$(i_1 + 1) + (i_2 + 1) + \cdots + (i_n + 1) = n + k - 1.$$

Therefore, the recursive solution is provided by the linear recursion:

$$L_1 = n, \quad L_k = (n + k - 1)L_{k-1}. \tag{13}$$

By eq. (13), we obtain the solution of the problem of rows in close form:

Proposition 5.1.

$$L_k = n(n+1)(n+2)\cdots(n+k-1),$$

that is the so called *rising factorial*

$$\langle n \rangle_k \stackrel{def}{=} n(n+1)(n+2)\cdots(n+k-1).$$

$\square$

5.2 k-multisets on n-sets and multiset coefficients

Let X be a finite set, $|X| = n$, say $X = \underline{n} = \{1, 2, \ldots, n\}$.

A *multiset* on the set $X = \{1, 2, \ldots, n\}$ is a *function*

$$\rho : \{1, 2, \ldots, n\} \to \mathbb{N}.$$

The value $\rho(i)$ is the *multiplicity* of the element $i \in \underline{n}$.

Notice that the notion of multiset is the modern and transparent formalization of *disposition with repetition* of the old fashioned "Combinatorial Calculus".

A multiset $\rho : \{1, 2, \ldots, n\} \to \mathbb{N}$ is called of cardinality k (k-multiset, for short) if

$$\sum_{k=1}^{n} \rho(i) = k.$$

Given $n, k \in \mathbb{N}$, the corresponding *multiset coefficient* is by definition:

$$\left\langle {n \atop k} \right\rangle \stackrel{def}{=} \#\ \text{k-multisets on an n-set}.$$

5.2.1 Dispositions with repetitions and nondecreasing words

Let $A = \{a_1 < a_2 < \ldots < a_n\}$ be an *alphabet* on n letters, that is a finite n-set endowed with a total order $<$.

A word of length k on A, say

$$w = a_{i_1} a_{i_2} \ldots a_{i_k}$$

is said to be *nondecreasing* whenever $a_{i_1} \leq a_{i_2} \leq \cdots \leq a_{i_k}$.

Clearly, given a nondecreasing word of length k on n letters, the function

$$\rho : A = \{a_1 < a_2 < \ldots < a_n\} \to \mathbb{N}.$$

such that

$$\rho(a_i) \ = \ \#\ \textit{of repetitions of the letter } a_i \textit{ in the word } w,$$

is a k-multiset on the n-set $A = \{a_1 < a_2 < \ldots < a_n\}$.

Conversely, given any k-multiset on the n-set $A = \{a_1 < a_2 < \ldots < a_n\}$ we can write its elements *with their multiplicities/repetitions* (in a unique way) in a nondecreasing order.

Then, the two families are *bijectively equivalent* and, a fortiori, they have the same cardinality $\left\langle {n \atop k} \right\rangle$. In the language of the old fashioned Combinatorial Calculus, increasing words are called *dispositions with repetitions*.

Example 5.2. Let $A = \{a_1 < a_2 < a_3\}$ and let

$$w = a_1 a_1 a_2 a_3 a_3 a_3$$

be a nondecreasing word of length 6. It bijectively corresponds to the 6-multiset ρ on $A = \{a_1 < a_2 < a_3\}$ such that

$$\rho(a_1) = 2, \ \rho(a_2) = 1, \ \rho(a_3) = 3.$$

$\square$

5.3 On the computation of multiset coefficients: close form formulae

We have a beautiful close form formula to compute multiset coefficients:

Proposition 5.3. *Let $n, k \in \mathbb{N}$. Then*

$$\left\langle {n \atop k} \right\rangle = \frac{\langle n \rangle_k}{k!}.$$

Proof. We shall use the overcounting principle. The k-multisets on the set $\underline{n}$ are *sheep*, the arrangements of k flags on n flagpoles are *legs*.

Indeed, given an arrangement of the k flags, with i_1 flags on flagpole 1, i_2 flags on flagpole 2, ..., i_n flags on flagpole n ($i_1 + i_2 + \cdots + i_n = k$) define a k-multiset on $\underline{n}$ by setting:

$$\rho : \underline{n} \to \mathbb{N},$$

$$\rho(1) = i_1, \quad \rho(2) = i_2, \quad \ldots, \rho(n) = i_n.$$

Notice that any k-multiset can be obtained in this way. But flag labels do not matter! Therefore, there are $k!$ different arrangements of flags that give rise to the same k-multiset. In a more formal way, we say that the above construction is a $k! \mapsto 1$ correspondence.

Then

$$\left\langle {n \atop k} \right\rangle = \frac{L_k}{k!} = \frac{\langle n \rangle_k}{k!},$$

by Proposition 5.1. $\square$

Remark 5.4. *Note that:*

$$\left\langle {n \atop k} \right\rangle = \frac{\langle n \rangle_k}{k!} = \frac{(n+k-1)!}{k!(n-1)!} = \binom{n+k-1}{k}.$$

$\square$

5.4 Multiset coefficients: recursive computation

By definition, we have:

$$\left\langle {0 \atop k} \right\rangle = \delta_{0,k}$$

and

$$\left\langle {n \atop 0} \right\rangle = 1.$$

Proposition 5.5. *Let $n, k \in \mathbb{N}$. Then*

$$\left\langle {n \atop k} \right\rangle = \sum_{i=0}^{k} \left\langle {n-1 \atop i} \right\rangle. \tag{14}$$

Proof. We shall use the bad element method. Fix the element $n \in \underline{n}$ and consider the disjoint/exhaustive cases:

$$\rho(n) = k;\ \textit{this case has cardinality} \left\langle {n-1 \atop 0} \right\rangle,$$

$$\rho(n) = k-1;\ \textit{this case has cardinality} \left\langle {n-1 \atop 1} \right\rangle,$$

$$\rho(n) = k-2;\ \textit{this case has cardinality} \left\langle {n-1 \atop 2} \right\rangle,$$

$$\vdots$$

$$\rho(n) = 0;\ \textit{this case has cardinality} \left\langle {n-1 \atop k} \right\rangle.$$

Then

$$\left\langle {n \atop k} \right\rangle = \left\langle {n-1 \atop 0} \right\rangle + \left\langle {n-1 \atop 1} \right\rangle + \left\langle {n-1 \atop 2} \right\rangle + \cdots + \left\langle {n-1 \atop k} \right\rangle.$$

□

The recursion (14) has (variable) step $k+1$. But, since

$$\sum_{i=0}^{k-1} \left\langle {n-1 \atop i} \right\rangle = \left\langle {n \atop k-1} \right\rangle,$$

it is equilavalent to the step 2 recursion

$$\left\langle {n \atop k} \right\rangle = \left\langle {n \atop k-1} \right\rangle + \left\langle {n-1 \atop k} \right\rangle. \tag{15}$$

□

5.5 The matrix of multiset coefficients as a recursive matrix

Consider the biinfinite matrix

$$M : \mathbb{N} \times \mathbb{N} \to \mathbb{R}, \quad M : (n,k) \mapsto M(n,k) \stackrel{def}{=} \left\langle {n \atop k} \right\rangle,$$

that is

$$M = \left[\left\langle {n \atop k} \right\rangle \right]_{n,k\in\mathbb{N}}.$$

By eq. (15), it is the matrix

$$\begin{array}{c|cccccccc}
 & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \cdots & \boldsymbol{k} & \cdots \\
\hline
\mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \cdots \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} & \cdots \\
\mathbf{2} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \cdots & \boldsymbol{k+1} & \cdots \\
\mathbf{3} & \mathbf{1} & \mathbf{3} & \mathbf{6} & \mathbf{10} & \mathbf{15} & \cdots & \left\langle {\mathbf{3} \atop \boldsymbol{k}} \right\rangle & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\boldsymbol{n} & \left\langle {\boldsymbol{n} \atop \mathbf{0}} \right\rangle & \left\langle {\boldsymbol{n} \atop \mathbf{1}} \right\rangle & \left\langle {\boldsymbol{n} \atop \mathbf{2}} \right\rangle & \left\langle {\boldsymbol{n} \atop \mathbf{3}} \right\rangle & \left\langle {\boldsymbol{n} \atop \mathbf{4}} \right\rangle & \cdots & \left\langle {\boldsymbol{n} \atop \boldsymbol{k}} \right\rangle & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}.$$

Since

$$\left\langle {0 \atop k} \right\rangle = \delta_{0k},$$

the 0-row generating series is:

$$M_0(t) = 1.$$

Since

$$\left\langle {1 \atop k} \right\rangle = 1 \; for \; every \; n \in \mathbb{N},$$

the 1-row generating series is:

$$M_1(t) = 1 + t + t^2 + t^3 + \cdots + t^k + \cdots. \tag{16}$$

Note that, since

$$(1-t)(1+t+t^2+t^3+\cdots+t^k+\cdots)=1$$

in the algebra $\mathbb{R}[[t]]$ of formal power series, we can consistently write:

$$M_1(t)=1+t+t^2+t^3+\cdots+t^k+\cdots=\frac{1}{1-t}.$$

Proposition 5.6. *The matrix*

$$M=\left[\left\langle{n \atop k}\right\rangle\right]_{n,k\in\mathbb{N}}$$

is a recursive matrix *having recursion rule*

$$M_1(t)=1+t+t^2+t^3+\cdots+t^k+\cdots=\frac{1}{1-t}.$$

Proof. We have to prove that:

$$M_n(t)\overset{def}{=}\sum_{k=0}^{\infty}\left\langle{n \atop k}\right\rangle t^k$$

equals

$$M_1(t)\cdot M_{n-1}(t),$$

where

$$M_{n-1}(t)\overset{def}{=}\sum_{k=0}^{\infty}\left\langle{n-1 \atop k}\right\rangle t^k.$$

To do so, we write

$$M_1(t)\cdot M_{n-1}(t)=\gamma(t)=\sum_{k=0}^{\infty}c_k\ t^k$$

where

$$c_k\overset{def}{=}\sum_{j=0}^{k}\left\langle{1 \atop i}\right\rangle\left\langle{n-1 \atop k-j}\right\rangle,$$

that equals

$$\sum_{i=0}^{k}\left\langle{n-1 \atop i}\right\rangle=\left\langle{n \atop k}\right\rangle.$$

Hence,

$$M_1(t)\cdot M_{n-1}(t)=\gamma(t)=M_n(t).$$

□

Thus we have the following *multiset* version of the *binomial theorem*:

Corollary 5.7. *Let $n \in \mathbb{Z}^+$. We have*

$$(1+t+t^2+t^3+\cdots+t^k+\cdots)^n = \frac{1}{(1-t)^n} = \sum_{k=0}^{\infty} \left\langle {n \atop k} \right\rangle t^k.$$

□

Furthermore, we have:

Corollary 5.8. ***(Vandermonde convolutions for multiset coefficients)*** *Let $i, j \in \mathbb{N}$ and $i + j = n$. Then*

$$\left\langle {n \atop k} \right\rangle \overset{thm}{=} \sum_{h=0}^{k} \left\langle {i \atop h} \right\rangle \left\langle {j \atop k-h} \right\rangle. \tag{17}$$

Proof. It immediately follows from Proposition 4.4, as a special case. □

5.6 A glimpse on combinatorial identities between binomial coefficients and multiset coefficients

Let $n, m \in \mathbb{N}$ and recall that

$$\alpha(t) = \sum_{k=0}^{\infty} \binom{n}{k} t^k \overset{thm}{=} (1+t)^n$$

and

$$\beta(t) = \sum_{k=0}^{\infty} \left\langle {m \atop k} \right\rangle t^k \overset{thm}{=} \frac{1}{(1-t)^m}.$$

Let

$$\alpha^*(t) \overset{def}{=} (1-t)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} t^k.$$

Consider the product series

$$\alpha^*(t)\beta(t) = \frac{(1-t)^n}{(1-t)^m} = \sum_{k=0}^{\infty} c_k t^k,$$

where

$$c_k \overset{def}{=} \sum_{h=0}^{k} (-1)^h \binom{n}{h} \left\langle {m \atop k-h} \right\rangle. \tag{18}$$

Proposition 5.9. *The value (18) equals:*

1. *If $n > m$ and $k > 0$, then*

$$\sum_{h=0}^{k} (-1)^h \binom{n}{h} \left\langle {m \atop k-h} \right\rangle =$$

$$= \binom{n-m}{k} (-1)^k.$$

2. If $n = m$ and $k > 0$, then

$$\sum_{h=0}^{k} (-1)^h \binom{n}{h} \left\langle {m \atop k-h} \right\rangle =$$

$$= 0.$$

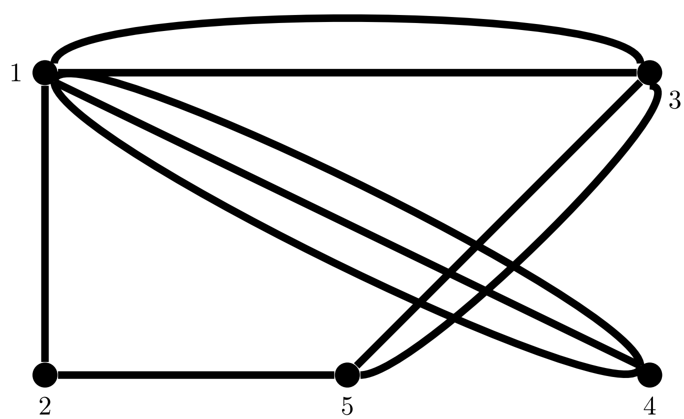
3. If $n < m$ and $k > 0$, then

$$\sum_{h=0}^{k} (-1)^h \binom{n}{h} \left\langle {m \atop k-h} \right\rangle =$$

$$= \left\langle {m-n \atop k} \right\rangle.$$

5.7 Multigraphs

A (labelled) *multigraph* on a set V of n *vertices* - say, $V = \{1, 2, \ldots, n\}$ - is, roughly speaking, a finite set of vertices V joined by *multiple edges.* Multiple edges that join the same pair of vertices are also called *parallel edges.*



Therefore, we formalize the notion of a multigraph G_ρ in the following way. A multigraph G_ρ is a pair

$$G_\rho = (V, E_\rho),$$

where $V = \{1, 2, \ldots, n\}$ is the set of vertices and the <u>multiset</u> E_ρ of of multiple edges is the multiset on the set of *non ordered* pairs of V:

$$E_\rho : \{A \subseteq V,\ |A| = 2\} \to \mathbb{N},$$

where

$$E_\rho(\ \{i, j\}\ ) \stackrel{def}{=} \#\ \text{parallel edges between i and j}, \quad i, j \in V.$$

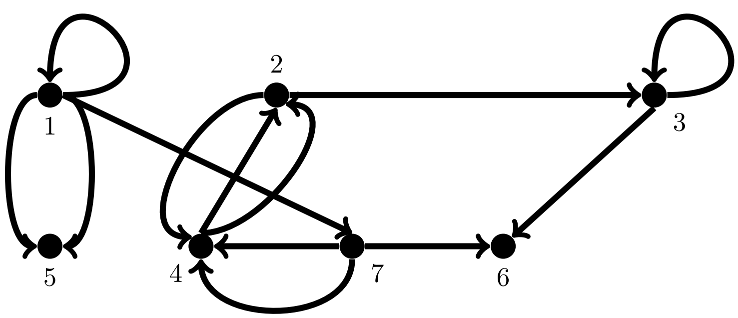
Proposition 5.10. *The number of multigraphs G_ρ on n vertices is $+\infty$.*

Proposition 5.11. *The number of multigraphs G_ρ on n vertices with exactly k edges is*

$$\left\langle \begin{matrix} \binom{n}{2} \\ k \end{matrix} \right\rangle.$$

5.8 Multidigraphs

A (labelled) *multidigraph* on a set V of n *vertices* - say, $V = \{1, 2, \ldots, n\}$ - is, roughly speaking, a finite set of vertices V joined by *multiple arrows.* Multiple arrows with the same direction that join the same *pair* of vertices are also called *parallel arrows.*



Therefore, we formalize the notion of a multidigraph $\vec{G}_\rho$ in the following way.

A multidigraph $\vec{G}_\rho$ is a pair

$$\vec{G}_\rho = (V, \vec{E}_\rho),$$

where $V = \{1, 2, \ldots, n\}$ is the set of vertices and the *multiset* $\vec{E}_\rho$ of multiple arrows is the multiset on the set of *ordered* pairs of V:

$$\vec{E}_\rho \colon V \times V \to \mathbb{N},$$

where

$$\vec{E}_\rho\ (\ (i,j)\ ) \stackrel{def}{=} \#\ \text{parallel arrows from i to j}, \quad (i,j) \in V \times V.$$

Proposition 5.12. *The number of multidigraphs* $\vec{G}_\rho$ *on* n *vertices is* $+\infty$*.*

Proposition 5.13. *The number of multidigraphs* $\vec{G}_\rho$ *on* n *vertices with exactly* k *arrows is*

$$\left\langle {n^2 \atop k} \right\rangle.$$

Proposition 5.14. *The number of multidigraphs* $\vec{G}_\rho$ *with no loops on* n *vertices with exactly* k *arrows is*

$$\left\langle {n(n-1) \atop k} \right\rangle.$$

6 Equations with natural integer solutions

6.1 The general case

Let $n, k \in \mathbb{Z}^+$. Consider the equation

$$x_1 + x_2 + \cdots + x_n \;=\; k. \tag{19}$$

A vector $(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n)$ such that

$$\underline{x}_1 + \underline{x}_2 + \cdots + \underline{x}_n \;=\; k, \quad \underline{x}_i \in \mathbb{N},\ i = 1, 2, \ldots, n, \tag{20}$$

is called a *nonnegative integer solution* of the equation (19).

Clearly, given a nonnegative integer solution (20) of (19), if we define

$$\rho : \underline{n} \to \mathbb{N}, \quad \rho(i) = \underline{x}_i,\ i = 1, 2, \ldots, n,$$

we get a k-multiset on an n-set, and vice versa.

Then,

Proposition 6.1. *The number of nonnegative integer solutions* (20) *of* (19) *is*

$$\left\langle {n \atop k} \right\rangle$$

□

A vector $(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n)$ such that

$$\underline{x}_1 + \underline{x}_2 + \cdots + \underline{x}_n \;=\; k, \quad \underline{x}_i \in \{0, 1\},\ i = 1, 2, \ldots, n, \tag{21}$$

is called a *binary solution* of the equation (19).

Clearly, given a binary solution (21) of (19), if we define

$$\rho : \underline{n} \to \{0, 1\}, \quad \rho(i) = \underline{x}_i,\ i = 1, 2, \ldots, n,$$

we get a k-subset of a n-set, and vice versa.

Then,

Proposition 6.2. *The number of binary solutions* (21) *of* (19) *is*

$$\binom{n}{k}$$

$\square$

6.2 The case subject to lower bounds

Let

$$(a_1, a_2, \ldots, a_n), \quad a_i \in \mathbb{N}, \ i = 1, 2, \ldots, n,$$

be a vector with natural integer entries.

Problem: how many nonnegative solutions $(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n)$ of (19), subject to the lower bounds

$$\underline{x}_1 \geq a_1, \ \underline{x}_2 \geq a_2, \ \ldots, \ \underline{x}_n \geq a_n$$

are there?

In order to answer this question, we may start by performing the following substitution of variables in equation (19):

$$z_i = x_i - a_i \iff x_i = z_i + a_i, \ i = 1, 2, \ldots, n,$$

and

$$\underline{x}_i \geq a_i \iff z_i \geq 0.$$

Thus, equation (19) becomes

$$z_1 + z_2 + \cdots + z_n \ = \ k - a_1 - a_2 - \cdots - a_n, \tag{22}$$

and the Problem reduces to:

How many nonnegative solutions of equation (22) are there?

Clearly, the answer is:

$$\left\langle {n \atop k - a_1 - a_2 - \cdots - a_n} \right\rangle. \tag{23}$$

6.3 The generalized Gergonne problem

Consider a linearly ordered n-set, say $\underline{n} \ = \ 1 < 2 < 3 < \cdots < n$ (for instance, a deck of playing cards).

Take at random a k-subset $S \subseteq \underline{n}$, say

$$S \ = \ i_1 < i_2 < \cdots < i_k. \tag{24}$$

Fix a third parameter $m \in \mathbb{Z}^+$, which is the so called *minimum lack parameter.*

The set (24) is said to be a *winning set* whenever:

$$i_{s+1} - i_s \geq m + 1, \quad s = 1, 2, \ldots, k - 1. \tag{25}$$

In plain words: between two consecutive elements i_s and i_{s+1} that belong to S, there are at least m elements of $\underline{n}$ that do not belong to S.

Clearly, such a winning k-subset S exists whenever

$$k + (k-1)m \leq n.$$

Problem: Given $n, k, m \in \mathbb{Z}^+$, compute the probability

$$\mathbf{P}_{n,k,m}$$

that a k-subset S (see (24)) is a winning set (see (25)). $\square$

Clearly,

$$\mathbf{P}_{n,k,m} \;=\; \frac{\mathbf{G}_{n,k,m}}{\binom{n}{k}},$$

where

$$\mathbf{G}_{n,k,m} \stackrel{def}{=} \#\ winning\ k-subsets.$$

Solution. Given a k-subset (24), consider the $(k+1)$-tuple

$$x_1,\ x_2, \ldots,\ x_k,\ x_{k+1}$$

where

$$x_1 = i_1 - 1,\ x_2 = i_2 - i_1 - 1, \ldots,\ x_k = i_k - i_{k-1} - 1,\ x_{k+1} = n - i_k. \quad (26)$$

Clearly, the k-subset (24) uniquely determines the $(k+1)$-tuple (26) such that

$$x_1 + x_2 + \cdots + x_k + x_{k+1} \;= n - k \quad (27)$$

and vice versa. Furthermore, the k-subset (24) is *winning* if and only if the $(k+1)$-tuple (26) is such that

$$x_1 \geq 0, \quad x_2 \geq m,\ \ldots, \quad x_k \geq m, \quad x_{k+1} \geq 0. \quad (28)$$

Therefore, *winning* k-subsets (24) satifying (25) bijectively correspond to the nonnegativite integer solutions of the equation (27), with lower bounds (28).

Equation (23) implies:

$$\mathbf{G}_{n,k,m} \;=\; \left\langle \begin{matrix} k+1 \\ n-k-(k-1)m \end{matrix} \right\rangle \;=\; \binom{n-mk+m}{k}. \quad (29)$$

Then

Proposition 6.3. *We have*

$$\mathbf{P}_{n,k,m} \;=\; \frac{\binom{n-mk+m}{k}}{\binom{n}{k}}.$$

$\square$

The classical Gergonne problem focuses on the case in which $m = 1$ i.e., no two consecutive playcards in the k-subset $S \;=\; i_1 < i_2 < \cdots < i_k$ can be *adjacent*. Then

$$\mathbf{G}_{n,k,1} \;=\; \binom{n-k+1}{k} \ \text{ and } \ \mathbf{P}_{n,k,1} \;=\; \frac{\binom{n-k+1}{k}}{\binom{n}{k}}. \quad (30)$$

7 Three statistics of Quantum Physics

7.1 The Bose-Einstein statistics

In quantum statistics, *Bose-Einstein (B–E) statistics* describe one of the two possible ways in which a collection of non-interacting, indistinguishable particles may occupy a set of available discrete energy states at thermodynamic equilibrium. The aggregation of particles in the same state is a characteristic of particles obeying Bose-Einstein statistics, accounts for the cohesive streaming of laser light and the frictionless creeping of superfluid helium. The theory of this behaviour was developed $(1924-25)$ by Satyendra Nath Bose, who recognized that a collection of identical and indistinguishable particles can be distributed in this way. The idea was later adopted and extended by Albert Einstein in collaboration with Bose.

The Bose-Einstein statistics apply only to those particles not limited to single occupancy of the same state, that is, particles that do not obey the *Pauli exclusion principle* restrictions. Such particles have integer values of spin and are named *bosons*, after the statistics that correctly describe their behaviour. There must also be no significant interaction between the particles.

Suppose we have a given number of energy levels, characterized by the index i, each having energy ε_i and containing a total of k_i particles. Suppose further that each level contains n_i distinct sub-levels, but all with the same energy and distinguishable from each other. For example, two particles could have different moments and consequently be distinguishable, but they could have the same energy. The value n_i at the i-th level is called *degeneration* of that energy level. *Any number of particles can occupy the same sublevel.*

For the sake of simplicity, let's write $k = k_i$ and $n = n_i$.

Let $w(k, n)$ be the number of ways to distribute k *indistinguishable* particles into n *distinguishable* sublevels of a certain energy level.

Theorem 7.1. *We have:*

$$w(k, n) = \left\langle {n \atop k} \right\rangle.$$

Proof. Clearly, (k, n)-BE distributions bijectively correspond to distributions of k *indistiguishable* balls (particles) into n *distiguishable* boxes (energy sublevels), that is to k-multisets on a n-set:

$$\rho : \underline{n} \to \mathbb{N}, \quad \sum_{i=1}^{n} \rho(i) = k,$$

where

$$\rho(i) \ = \ \#\ of\ balls\ in\ the\ box\ i, \quad i = 1, 2, \ldots, n. \tag{31}$$

$\square$

7.2 The Fermi-Dirac statistics

In quantum statistics, the *Fermi-Dirac (F–D) statistics* describe a distribution of particles over energy states in systems consisting of many identical particles that obey the *Pauli exclusion principle*. It is named after Enrico Fermi and Paul Dirac, each of whom discovered the method independently (although Fermi defined the statistics earlier than Dirac).

Fermi-Dirac (FD) statistics apply to identical particles with half-integer spin in a system with thermodynamic equilibrium. Additionally, the particles in this system are assumed to have negligible mutual interaction. That allows the multi-particle system to be described in terms of single-particle energy states. The result is the FD distribution of particles over these states which includes the condition that *no two particles can occupy the same state* (*Pauli exclusion principle*); this has a considerable effect on the properties of the system. Since FD statistics apply to particles with half-integer spin, these particles have come to be called *fermions*. It is most commonly applied to *electrons*, a type of fermion with spin 1/2.

Let $w^*(k,n)$ be the number of ways to distribute k *indistinguishable* particles into n *distinguishable* sublevels of a certain energy level obeying the condition that *no two particles can occupy the same state* (Pauli exclusion principle).

Theorem 7.2. *We have:*

$$w^*(k,n) = \binom{n}{k}.$$

Proof. Clearly, this case is the special case of (31) obeying the condition that *no two particles can occupy the same state* (Pauli exclusion principle):

$$\rho(i) \;=\; \#\; of\ balls\ in\ the\ box\ i, \quad \rho(i) \in \{0,1\}, \quad i = 1,2,\ldots,n.$$

Then, (k,n)-FD distributions bijectively correspond to k-subsets on a n-set. □

7.3 The Giovanni Gentile jr statistics

Giovanni Gentile jr was born in Napoli in 1906. In 1937 he participated in the **Fisica Teorica** concorso held by the University of Palermo. Previously only one concorso for this subject had been held in Italy - the one won by Enrico Fermi, Enrico Persico and Aldo Pontremoli - and there were many worthy scholars in Italy, some of whom had to be necessarily sacrificed. Ettore Majorana, who had retired in almost complete isolation, presented his candidacy surprising everyone and upsetting the agreement established among the commissioners (who were E. Fermi, O. Lazzarino, E. Persico, G. Polvani, A. Carrelli). The concorso was suspended for a few months and Majorana was appointed professor for *exceptional merits*. When the competition resumed, a triad of winners was announced: Giancarlo Wick, Giulio Racah and Giovanni Gentile. As soon as he was proclaimed among the winners, Gentile was called to hold the chair of Fisica Teorica in Milan.

His major theoretical contribution is constituted by the memoirs on *intermediate statistics*. In the two types of statistics for atomic objects, Bose-Einstein and Fermi-Dirac, the maximum number of occupations for each cell of the phase space is either infinite or one, respectively. Gentile began to deal with the more general case that the maximum number of occupations was any integer greater than one, establishing the general energy distribution formulas. These formulas are applied to the case of degenerate gases, in which the maximum number of occupancy is at most that of the molecules making up the gas - therefore, they are not tractable with Bose-Einstein statistics - obtaining a theoretical treatment of the Bose-Einstein gas condensation phenomenon and an interpretation of some singular properties of liquid helium. From these works began a theoretical research sector dedicated to the treatment of particles subject to intermediate statistics called, in honor of Gentile, *gentilioni*, distinct from *bosons* and *fermions*.

An attack of septicemia, a consequence of a banal dental abscess, killed him in Milan on 30 March 1942.

Given a positive integer $p \in \mathbb{Z}^+$, let $c^p(n,k)$ be the number of ways to distribute k *indistinguishable* particles into n *distinguishable* sublevels of a certain energy level obeying the condition that *not more* than p particles can occupy the same state (p the *parameter* of the statistics).

Clearly, if $p = 1$, then we get the Fermi-Dirac statistics $w^*(k,n) = \binom{n}{k}$ and, if $p \to \infty$, then we get the Bose-Einstein statistics $w(k,n) = \left\langle {n \atop k} \right\rangle$. Therefore, the Gentile coefficients $c^p(n,k)$ provide a common generalization/unification of both *binomial* and *multiset* coefficients.

First, we get a computation of the $c^p(n,k)$'s by recursion. Again, the particles are thougt as k *indistinguishable* balls to be distributed into n *distinguishable* boxes.

Clearly,

$$c^p(0,k) = \delta_{0,k}, \quad c^p(n,0) = 1 \ for \ every \ n \in \mathbb{N}.$$

Furthermore, from the definitions, it follows:

$$c^p(n,k) = 0 \ whenever \ k > np.$$

Proposition 7.3. *We have:*

$$c^p(n,k) = \sum_{i=0}^{p} c^p(n-1,k-i).$$

Proof. Given a Gentile distribution of parameter $p \in \mathbb{Z}^+$, denote with

$$\rho(i) = \# \ of \ balls \ in \ the \ box \ i,$$

for $i = 1, 2, \ldots, n$.

We shall use the bad element method. Let the last box (with label n) be the bad element. We have the exhaustive/disjoint cases;

$$\rho(n) = 0, \ having \ cardinality \ c^p(n-1,k),$$

$$\rho(n) = 1, \ \textit{having cardinality } c^p(n-1, k-1),$$

$$\vdots$$

$$\rho(n) = p, \ \textit{having cardinality } c^p(n-1, k-p).$$

Then,

$$c^p(n,k) = \ c^p(n-1,k) + c^p(n-1,k-1) + \cdots + c^p(n-1,k-p).$$

□

Given $p \in \mathbb{Z}^+$, consider the biinfinite matrix

$$M^{(p)} : \mathbb{N} \times \mathbb{N} \to \mathbb{R}, \quad M^{(p)} : (n,k) \mapsto M(n,k) \stackrel{def}{=} c^p(n,k),$$

that is

$$M^{(p)} = [\ c^p(n,k)\ ]_{n,k \in \mathbb{N}} \, .$$

For example, if $p = 2$, the matrix $M^{(2)}$ is:

$$\begin{array}{c|cccccccc}
 & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \cdots & \boldsymbol{k} & \cdots \\
\hline
\mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \cdots \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \cdots \\
\mathbf{2} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{2} & \mathbf{1} & \cdots & \mathbf{0} & \cdots \\
\mathbf{3} & \mathbf{1} & \mathbf{3} & \mathbf{6} & \mathbf{7} & \mathbf{6} & \cdots & \boldsymbol{c^2(3,k)} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\boldsymbol{n} & \boldsymbol{c^2(n,0)} & \cdots & \cdots & \cdots & \cdots & \cdots & \boldsymbol{c^2(n,k)} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array} .$$

Proposition 7.4. *Given $p \in \mathbb{Z}^+$, the Gentile matrix $M^{(p)}$ (of parameter p) is a* recursive matrix *having recursion rule*

$$M_1^{(p)}(t) = 1 + t + t^2 + \cdots + t^p.$$

Proof. Consider the product series

$$M_1^{(p)}(t)M_{n-1}^{(p)}(t) = (1 + t + t^2 + \cdots + t^p) \cdot (\sum_{k=0}^{\infty} c^p(n-1,k)t^k),$$

and write

$$M_1^{(p)}(t)M_{n-1}^{(p)}(t) = \gamma(t) = \sum_{k=0}^{\infty} c_k t^k$$

with

$$c_k = \sum_{i=0}^{k} c^p(1,i)c^p(n-1,k-i) = \sum_{i=0}^{p} c^p(n-1,k-i) = c^p(n,k),$$

by Proposition 7.3. Then $\gamma(t) = \sum_{k=0}^{np} c^p(n,k)t^k = M_n^{(p)}(t)$. □

Hence, we have the following generalization of the *binomial theorem*:

Corollary 7.5. *Given $p \in \mathbb{Z}^+$, we have:*

$$(1 + t + t + t^2 + \cdots + t^p)^n = \sum_{k=0}^{np} c^p(n,k)t^k.$$

□

8 Compositions of finite sets and multinomial coefficients

8.1 The type of a composition

Given a finite set X, say $X = \underline{n} = \{1, 2, \ldots, n\}$, a $k-$*composition* of $\underline{n} = \{1, 2, \ldots, n\}$ is an *ordered* $k-$*tuple*:

$$(A_1, A_2, \ldots, A_k), \quad A_i \subseteq \underline{n} = \{1, 2, \ldots, n\}, \ i = 1, 2, \ldots, k, \tag{32}$$

such that

1. if $i \neq j$, then $A_i \cap A_j = \emptyset$;
2. $\cup_{i=1}^{k} A_i = \underline{n}$.

The elements A_i are called *blocks* of the composition.
The *type* of the composition (32) is the *ordered* $k-$*tuple* of natural integers:

$$(h_1 = |A_1|, h_2 = |A_2|, \ldots, h_k = |A_k|). \tag{33}$$

8.2 Multinomial coefficients

Given $n \in \mathbb{N}$ and an *ordered* $k-tuple$ of natural integers $(h_1, h_2, \cdots, h_k)$, define the *multinomial coefficients* in the following way:

$$\binom{n}{h_1, h_2, \ldots, h_k} \stackrel{def}{=} \# \text{ k-compositions of an n-set of type } (h_1, h_2, \cdots, h_k). \tag{34}$$

Clearly,

$$\binom{n}{h_1, h_2, \ldots, h_k} \neq 0$$

if and only if

$$h_1 + h_2 + \cdots + h_k = n.$$

We can compute the *multinomial coefficients* by means of the following close form formula:

Proposition 8.1. *Let $h_1 + h_2 + \cdots + h_k = n$. Then, we have:*

$$\binom{n}{h_1, h_2, \ldots, h_k} = \frac{n!}{h_1!, h_2!, \cdots, h_k!}.$$

Proof. We shall use the shepherd's principle. The sheep are k-compositions of type $(h_1, h_2, \cdots, h_k)$ that are, in turn, *bijectively equivalent* to distributions of n (distinguishable) balls (labelled: $1, 2, \ldots, n$) into k (distinguishable) boxes (labelled: $1, 2, \ldots, k$), subject to the conditions: *there are exactly* h_i balls located in the box $i = 1, 2, \ldots, k$. The set of balls located in the box i, $i = 1, 2, \ldots, k$, is the ith block A_i of the composition (32), and vice versa. Note that in *Probability Theory* and *Quantum Mechanics* these numbers h_i are called *occupancy numbers*.

Now, suppose that the n balls are turned into *books* and the boxes are turned into *shelves*, with h_i available positions. Since $h_1 + h_2 + \cdots + h_k = n$ (the total numer of available positions), the number of ways to distribute n books into these k shelves (with a total numer of available positions $h_1 + h_2 + \cdots + h_k = n$) is exactly the number of permutations $n!$ of the n objects (these are the legs).

Now, turn the books back into balls and the shelves back into boxes. Then, the order of the objects located in position i doesn't matter! Thus, there are exactly $h_1! h_2! \cdots, h_k!$ legs for each sheep. Hence,

$$\binom{n}{h_1, h_2, \ldots, h_k} = \frac{n!}{h_1!, h_2!, \cdots, h_k!}.$$

□

Remark 8.2. Directly from the combinatorial definition (34) *we infer the remarkable identity:*

$$\sum_{h_1, h_2, \ldots, h_k} \binom{n}{h_1, h_2, \ldots, h_k} = k^n. \tag{35}$$

From the combinatorial identity (35), we get

Corollary 8.3. *We have:*

$$\sum_{h_1,h_2,\ldots,h_k} \frac{n!}{h_1!, h_2!, \cdots, h_k!} = k^n.$$

We have the following *multinomial* version of the *binomial theorem*:

Corollary 8.4. *We have:*

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{(h_1,h_2,\ldots,h_k)} \binom{n}{h_1, h_2, \ldots, h_k} x_1^{h_1} x_2^{h_2} \cdots x_k^{h_k}.$$

□

9 Equivalence relations and partitions

Let $R \subseteq X \times X$ be binary relation a set X; as usual, we write xRx' to mean $(x, x') \in R$.

The relation R is said to be an *equivalence relation* whenever it satisfies the following properties:

1. xRx (reflexivity);
2. if xRy, then yRx (symmetry);
3. if xRy and yRz, then xRz (transitivity).

Given an element $x \in X$, its *equivalence class* is the subset

$$[x]_R \stackrel{def}{=} \{ \ y \in X; \ xRy \ \} \subseteq X.$$

We recall that $[x]_R = [x']_R$ if and only if xRx'. This implies

Proposition 9.1. *Equivalence classes are* pairwise disjoint*. Furthermore, they are nonempty and their union equals X.*

□

A *partition* of the set X is a subset

$$\Pi = \{ \ A_i \subseteq X; \ i \in I \ \} \subseteq \mathbb{P}(X)$$

such that

1. $A_i \neq \emptyset$;
2. if $i \neq j$, then $A_i \cap A_j = \emptyset$;
3. $\cup_i \ A_i = X$.

The elements $A_i \in \Pi$ are called *blocks* of the partition Π.

9.1 Bijections

Let

$$\mathbf{Rel}(X) = \{\ R;\ R\ equiv.\ rel.\ on\ X\ \}$$

be the set of all equivalence relations on X, and let

$$\mathbf{Par}(X) = \{\ \Pi;\ \Pi\ partition\ of\ X\ \}$$

be the set of all partitions of X.

Let

$$\mathbf{Rel}(X) \xrightarrow{C_1} \mathbf{Par}(X)$$

be the function

$$C_1 : R \mapsto \Pi_R,$$

where Π_R is the partition of X whose blocks are the equivalence classes of R.

Let

$$\mathbf{Par}(X) \xrightarrow{C_2} \mathbf{Rel}(X)$$

be the function

$$C_2 : \Pi \mapsto R_\Pi,$$

where R_Π is the equivalence relation on X where $xR_\Pi x'$ if and only if $x, x' \in X$ belong to the same block of Π.

But

$$R \overset{C_1}{\mapsto} \Pi_R \overset{C_2}{\mapsto} R_{\Pi_R} = R,$$

and

$$\Pi \overset{C_2}{\mapsto} R_\Pi \overset{C_1}{\mapsto} \Pi_{R_\Pi} = \Pi.$$

Then C_1 and C_2 are inverse functions and, therefore, they are *bijections.*

We summarize these facts by saying that $\mathbf{Rel}(X)$ and $\mathbf{Par}(X)$ are different but *bijectively equivalent* sets.

9.2 The Stirling numbers of the 2nd kind $S(n, k)$

Let $n, k \in \mathbb{N}$. Given a finite set $X = \{1, 2, \ldots, n\}$, a partition Π of $X = \underline{n}$ is said to be a k-partition whenever it has exactly k blocks.

Let

$$S(n, k) \overset{def}{=} \#\ \text{k-partitions of an n-set}.$$

The natural integers $S(n, k)$ are called *Stirling numbers of the 2nd kind.*

Notice that $S(0, 0) = 1$: the unique 0-partition of the 0-set $\emptyset$ is the empty partition, that is $\emptyset$ is the unique partition of itself.

Clearly, we have:

$$S(0, k) = \delta_{0k}, \quad S(n, 0) = \delta_{n0}.$$

We can compute the numbers $S(n, k)$ by means of the following recursion:

Proposition 9.2. *We have:*

$$S(n,k) = S(n-1,k-1) + k\ S(n-1,k).$$

Proof. We shall use the bad element method. Let $n \in \underline{n}$ be the bad element. We have two disjoint/exhaustive cases:

i) The singleton $\{n\}$ is a block of the partition. Then, to exhibit a k-partition of $\underline{n}$ is equivalent to exhibiting a $(k-1)$-partition of $\underline{n-1}$. Thus, the contribution of case i) is: $S(n-1,k-1)$.

ii) The singleton $\{n\}$ is not a block of the partition, that is the bad element n will stay in a block together with other elements of the set $\underline{n}$. We can construct partitions of this type by the following procedure: first, exhibit a k-partition of $\underline{n-1}$ (this can be done in $S(n-1,k)$ ways), then *insert* the bad element n into one of the k blocks (this can be done in k ways). Thus, the contribution of this case is: $kS(n-1,k)$. $\square$

Hence, we can compute the entries of the biinfinite matrix

$$M = [\ S(n,k)\ ]_{n,k\in\mathbb{N}}:$$

$$\begin{array}{c|cccccccc}
 & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \cdots & \boldsymbol{k} & \cdots \\
\hline
\mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \cdots \\
\mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \cdots \\
\mathbf{2} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \cdots \\
\mathbf{3} & \mathbf{0} & \mathbf{1} & \mathbf{3} & \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} & \cdots \\
\mathbf{4} & \mathbf{0} & \mathbf{1} & \mathbf{7} & \mathbf{6} & \mathbf{1} & \cdots & \mathbf{0} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\boldsymbol{n} & \boldsymbol{S(n,0)} & \cdots & \cdots & \cdots & \cdots & \cdots & \boldsymbol{S(n,k)} & \cdots
\end{array}.$$

9.3 The Bell numbers B_n

The *Bell numbers* B_n are defined in the following way:

$$B_n \stackrel{def}{=} \# \text{ partitions of an n-set.}$$

Note that, by definition

$$B_0 = 1, \quad B_1 = 1 :$$

the unique partition of $\emptyset$ is $\emptyset$, and the unique partition of the singleton set $\{1\}$ is the (singleton) set of blocks $\{ \{1\} \}$.

We can compute the Bell numbers B_{n+1} by means of the *Aitken recursion*:

Proposition 9.3. *Let $n \in \mathbb{Z}^+$. Then*

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

Proof. We shall use the bad element method. Let $n+1 \in \underline{n+1}$ be the bad element. We have $n+1$ disjoint/exhaustive cases: we classify the cases by means of the cardinality of the block B that contains the bad element $n+1$. Clearly, it may happen:

$$|B| = n - k + 1, \quad k = 0, 1, \ldots, n.$$

Now, fix $k = 0, 1, \ldots, n$. The block B can be uniquely expressed in the form:

$$B = B' \,\dot{\cup}\, \{n\},$$

where

$$|B'| = n - k, \quad B' \subseteq \underline{n}.$$

The subset B' can be chosen in

$$\binom{n}{n-k} = \binom{n}{k}$$

different ways.

All that is left is to partition the remaining k elements in $\underline{n+1} - B$: this can be done, by definition, in B_k ways: thus, the contribution of this case (fixed $k = 0, 1, \ldots, n$) is

$$\binom{n}{k} B_k.$$

By summing over all the possible values of k, we get:

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

$\square$

Example 9.4. We have:

$$B_0 = 1,\ B_1 = 1,\ B_2 = 2,\ B_3 = 5,\ B_4 = 15,\ B_5 = 52,\ B_6 = 203,$$

$$B_7 = 887,\ B_8 = 4140,\ B_9 = 21147,\ \dots$$

□

9.4 The Faà di Bruno coefficients

9.4.1 The type of a partition

Given a partition Π of a finite n-set, we say that Π has *type*

$$1^{\nu_1} 2^{\nu_2} 3^{\nu_3} \cdots n^{\nu_n}$$

whenever

$$\Pi \ has \ \nu_i \ blocks \ of \ cardinality \ i, \ for \ i = 1, 2, \dots, n.$$

The Faà di Bruno coefficients $P(n; 1^{\nu_1} 2^{\nu_2} \cdots)$ are defined in the following way:

$$P(n; 1^{\nu_1} 2^{\nu_2} \cdots) \stackrel{def}{=} \# \text{ partitions of type } 1^{\nu_1} 2^{\nu_2} 3^{\nu_3} \cdots \text{ of an n-set.}$$

Clearly,

$$P(n; 1^{\nu_1} 2^{\nu_2} \cdots) \neq 0$$

if and only if

$$\nu_1 + 2\nu_2 + 3\nu_3 + \cdots = n.$$

We can compute the Faà di Bruno coefficients by means of the remarkable close form formula:

Proposition 9.5. *If* $\nu_1 + 2\nu_2 + 3\nu_3 + \cdots = n$*, then*

$$P(n; 1^{\nu_1} 2^{\nu_2} \cdots) = \frac{n!}{(1!)^{\nu_1} (2!)^{\nu_2} (3!)^{\nu_3} \cdots} \cdot \frac{1}{\nu_1! \nu_2! \nu_3! \dots}.$$

Proof. We shall use the shepherd's principle. Partitions of type $1^{\nu_1} 2^{\nu_2} \cdots$ are the sheep while the legs are the compositions (of the same n-set) of type:

$$\left(1, \overset{\nu_1 \ times}{1}, 1, \cdots 1, 2, \overset{\nu_2 \ times}{2}, 2, \cdots 2, 3, \overset{\nu_3 \ times}{3}, 3, \cdots 3, \dots \right). \tag{36}$$

Given any composition of this type, if we neglect the order (i.e., we pass from ordered k-tuples to *sets*), we get a *unique* partition of type $1^{\nu_1} 2^{\nu_2} 3^{\nu_3} \cdots$.

But, in compositions of type (36), we can permute the blocks of the same cardinality without affecting neither the type of the composition nor the partition we produce. Then, there are precisely

$$\nu_1! \nu_2! \nu_3! \cdots$$

legs per each sheep. Therefore,

$$P(n;1^{\nu_1}2^{\nu_2}\cdots)$$

equals

$$\binom{n}{1,1,1,\dots,2,2,2,\dots,3,3,3,\dots}\cdot\frac{1}{\nu_1!\nu_2!\nu_3!\dots}$$

which in turn, equals

$$\frac{n!}{(1!)^{\nu_1}(2!)^{\nu_2}(3!)^{\nu_3}\cdots}\cdot\frac{1}{\nu_1!\nu_2!\nu_3!\dots}.$$

□

9.4.2 The n-th derivative of a composite function $f(g(t))$

Let $f,g:(a,b)\subseteq\mathbb{R}\to\mathbb{R}$ be functions of class $C^{(\infty)}_{(a,b)}$, and consider the composite function

$$(f\circ g)(t)=f(g(t)),\quad t\in(a,b).$$

Given $n\in\mathbb{Z}^+$, the nth derivative of the composite function $(f\circ g)(t)$ is explicitly provided by the *Faà di Bruno formula* (1855):

$$(f\circ g)^{(n)}(t)=\sum_{(\nu_1,\nu_2,\dots)}P(n;1^{\nu_1}2^{\nu_2}\cdots)\;f^{(|\nu|)}(g(t))\cdot(g^{(1)}(t))^{\nu_1}(g^{(2)}(t))^{\nu_2}\cdots,$$

where $|\nu|=\nu_1+\nu_2+\cdots$.

9.5 A concluding remark on partition statistics

From their combinatorial definitions, we immediately infer

Proposition 9.6. *Let $n\in\mathbb{N}$. Then*

$$B_n\;=\;\sum_{k=0}^{n}S(n,k)\;=\;\sum_{(\nu_1,\nu_2,\dots)}P(n;1^{\nu_1}2^{\nu_2}\cdots).$$

10 Permutations

10.1 Permutations and permutation digraphs

An *n-permutation* σ of an n-set (say, $\underline{n}=\{1,2,\dots,n\}$) is a bijection

$$\sigma:\underline{n}\longleftrightarrow\underline{n}$$

from the set onto itself. A permutation is usually described by its functional presentation, that is in the form

$$\sigma=\begin{pmatrix}1&2&3&\dots&n\\ \sigma(1)&\sigma(2)&\sigma(3)&\dots&\sigma(n)\end{pmatrix}.\tag{37}$$

Clearly, the number of *n-permutations* is: $n!$.

An *n-permutation digraph* is a digraph on n vertices (say, $V = \underline{n}$)

$$\vec{G} = (V, \vec{E})$$

such that, for every vertex $i \in V = \underline{n}$, there is a unique *arrow* with *head* i and a unique *arrow* with *tail* i.

Given the permutation (37), the *associated* n-permutation digraph is the digraph

$$\vec{G}_\sigma = (V = \underline{n}, \vec{E})$$

such that, for every vertex $i \in V = \underline{n}$, the unique arrow with head i is $i \to \sigma(i)$ and the unique arrow with tail i is $\sigma^{-1}(i) \to i$.

Example 10.1. If

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 7 & 4 & 6 & 1 & 3 & 8 & 2 \end{pmatrix}, \tag{38}$$

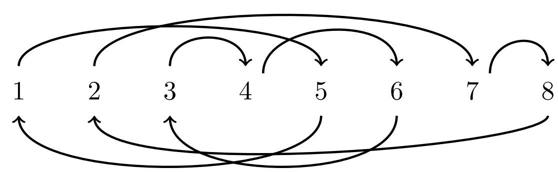
then

$$\vec{G}_\sigma \tag{39}$$

is the digraph on vertices $\{1, 2, \ldots, n\}$ whose arrows are

$$1 \to 5,\ 5 \to 1,\ 2 \to 7,\ 7 \to 8,\ 8 \to 2,\ 3 \to 4,\ 4 \to 6,\ 6 \to 3,$$

that is



□

We have:

Proposition 10.2. *The map*

$$\sigma \mapsto \vec{G}_\sigma$$

is a bijection *from the family of* n*-permutations to the family of* n*-permutation digraphs.*

10.2 Cycles and cyclic digraphs

Given a digraph $\vec{G} = (V, \vec{E})$, an *(oriented) path* (from the vertex i_1 to the vertex i_k) is a finite sequence of arrows

$$i_1 \to i_2 \to i_3 \to \cdots\cdots \to i_{h-1} \to i_h \to \cdots\cdots \to i_{k-2} \to i_{k-1} \to i_k.$$

A permutation k-digraph $\vec{G_\tau} = (V = \underline{k}, \vec{E})$ is said to be a *cyclic digraph* whenever it is *path connected*, that is, for every $i, j \in \underline{n}$, there exists a (unique) oriented path from the vertex i to the vertex j and there exists a (unique) oriented path from the vertex j to the vertex i.

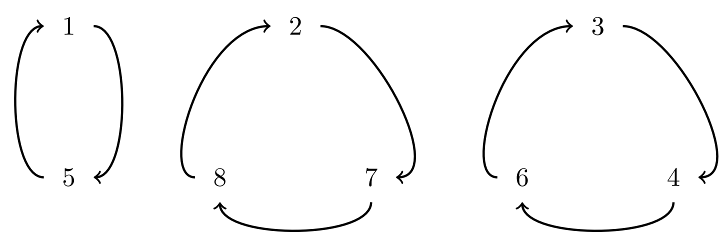
The permutation τ associated to a cyclic k-digraph $\vec{G_\tau}$ is said to be a *k-cycle.* Clearly, a k-cycle τ is a permutation of the form

$$\tau(i_k) = \tau(\tau(i_{k-1}) = \tau(\tau^{k-1}(i_1) = \tau^k(i_1) = i_1.$$

Example 10.3. In the permutation (38), we have three (sub)permutations C_1, C_2, C_3 which are cycles, namely:

$$C_1 = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 2 & 7 & 8 \\ 7 & 8 & 2 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 3 & 4 & 6 \\ 4 & 6 & 3 \end{pmatrix}.$$

The associated permutation digraphs are



whose *disjoint union* is the *same* as the graph of Example 10.1 (just different drawing).

□

Proposition 10.4. *The number of cycles on k elements (i.e., the number of cyclic digraphs on k vertices) equals* $(k-1)!$

Proof. We shall use the (inverse) shepherd's principle. Given a cycle τ and fixed element $i \in \underline{k}$, define the n-permutation

$$\sigma_i = \begin{pmatrix} 1 & 2 & 3 & \dots & k \\ \sigma(1) & \sigma(2) & \sigma(3) & \dots & \sigma(k) \end{pmatrix},$$

as

$$\sigma_i(1) = \tau(i+1),\ \sigma_i(2) = \tau(i+2), \dots, \sigma_i(h) = \tau(i+h), \dots,$$
$$\dots,\ \sigma_i(k) = \tau(i+k), \dots, \sigma_i(1) = \tau(i+k+1), \dots,$$

for $h = 0, 1, \dots, k$.

This way, we are able to produce *all* the *distinct* k-permutations of $\underline{k}$. Overall, these are $k!$.

But, depending from the choice of $i = 0, 1, \dots k-1$, there are are exactly k different k-permutations (legs) that are produced by one and the same cycle τ (sheep). Then, the number of cycles on k elements equals

$$\frac{k!}{k} = (k-1)!$$

$\square$

Example 10.5. Consider the 4-cycle τ such that

$$\tau(1) = 3,\ \tau(3) = 2,\ \tau(2) = 4,\ \tau(4) = 1.$$

The corresponding 4-permutations in the above construction are:

$$\sigma_0 = \begin{pmatrix} 1\,2\,3\,4 \\ 3\,2\,4\,1 \end{pmatrix},$$

$$\sigma_1 = \begin{pmatrix} 1\,2\,3\,4 \\ 2\,4\,1\,3 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 1\,2\,3\,4 \\ 4\,1\,3\,2 \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} 1\,2\,3\,4 \\ 1\,3\,2\,4 \end{pmatrix}.$$

Therefore, the number of 4-cycles is:

$$\frac{4!}{4} = 3! = 6.$$

$\square$

Given a k-*cycle* τ, any k-word of the form

$$(\tau(i), \tau(i+1), \ldots, \tau(k), \ldots, \tau(i-1)), \quad i = 1, 2, \ldots, k$$

is called a *word presentation* of the cycle τ.

Example 10.6. Given the 4-cycle τ of Ex. 10.5, its word presentations are

$$3241, \quad 2413, \quad 4132, \quad 1324.$$

$\square$

Clearly, any k-*cycle* admits k different word presentations.

10.3 The unique factorization of a permutation into disjoint cycles

To begin with, we state and proof this basic result in the language of *permutation digraph.*

Proposition 10.7. *Any permutation digraph* $\vec{G}_\sigma = (V, \vec{E})$ *is a* disjoint union *of cyclic digraphs. In symbols*

$$\vec{G}_\sigma = \vec{G}_{C_1} \,\dot\cup\, \vec{G}_{C_2} \,\dot\cup \cdots\cdots \,\dot\cup\, \vec{G}_{C_k},$$

where $\vec{G}_{C_1}, \vec{G}_{C_2}, \ldots, \vec{G}_{C_k}$ *are cyclic digraphs on disjoint subsets* $C_1, C_2, \ldots, C_k \subseteq V$ *of the vertex set of* $\vec{G}_\sigma$.

Proof. First, we prove that a permutation graph can be described as a union of cyclic digraphs, that is any vertex belongs to at least one cyclic subdigraph. Consider the following procedure/algorithm. Choose a vertex, say 1, and examine the unique arrow $1 \to \sigma(1)$ with head 1. We may have two cases.

1. If $1 = \sigma(1)$, then the arrow is indeed a loop (1-cycle) and we pass to examine a further vertex.
2. If $1 \neq \sigma(1)$, then $\sigma(1) \neq \sigma(\sigma(1))$ and so on, that is, every arrow we are producing will have different head and tail. Cleary, we can repeat this procedure over and over again. Yet, is it possible that at any step we will find new vertices? The answer is no, since the set of vertices is FINITE! This implies that, after a finite number of repetitions, we will find one among the vertices already produced, which must be the initial vertex 1.

We repeat this procedure for remaining vertices, if there are any left. Indeed, we proved that our permutation digraph is "covered" by cyclic subdigraphs.

Have these cyclic subdigraphs disjoint sets of vertices? If not, there must be at least one vertex that should be either head or tail of more than one arrow, which is a contradiction.

□

In the language of permutations, Proposition 10.7 reads:

Proposition 10.8. *Any permutation σ can be* uniquely factorized *into the product of disjoint cycles.*

10.4 The coefficients $C(n, k)$

Let $n, k \in \mathbb{N}$. The numbers

$$C(n, k)$$

are defined in the following way:

$$C(n, k) \stackrel{def}{=} \# \text{ n-permutations with k cycles.}$$

Clearly, we have

$$C(0, k) = \delta_{0k}, \quad C(n, 0) = \delta_{n0}.$$

We can compute the $C(n, k)$'s by means of the following recursion:

Proposition 10.9. *Let $n, k \in \mathbb{Z}^+$. Then,*

$$C(n, k) = C(n-1, k-1) + (n-1)C(n-1, k).$$

Proof. We shall use the bad element method. Fix $n \in \underline{n}$ as bad element.

We have two cases:

1. n is a fixed point, that is, the arrow $n \to n$ is indeed a loop.

   Therefore, the total number of this case is:

$$C(n-1, k-1).$$

2. n isn't a fixed point.

   Given such a permutation σ witk k cycles, consider the associated permutation digraph $\vec{G}_\sigma$. This permutation digraph is such that the (unique) cycle containing n has at least 2 vertices. How can we construct these permutation digraphs? At first, we have to costruct a permutation digraph $\vec{G^*}$ on the first $n-1$ elements $\underline{n-1}$ with k cycles: this can be done in $C(n-1, k)$ different ways.

   Now, we must insert the bad element n. This can be done putting n on any existing arrow of $\vec{G^*}$, in order to split it into a pair of two consecutive arrows. But $\vec{G^*}$ is a permutation digraph on $n-1$ vertices. Thus, the number of arrows is $n-1$. Hence, the insertion of the bad element n can be performed in $n-1$ different ways. Therefore, the total number of this second case is:

$$(n-1)C(n-1, k).$$

Thus,

$$C(n, k) = C(n-1, k-1) + (n-1)C(n-1, k).$$

□

Hence, we can compute the entries of the biinfinite matrix

$$M = [\ C(n, k)\ ]_{n,k \in \mathbb{N}}:$$

	0	**1**	**2**	**3**	**4**	$\cdots$	$\boldsymbol{k}$	$\cdots$
0	**1**	**0**	**0**	**0**	**0**	$\cdots$	**0**	$\cdots$
1	**0**	**1**	**0**	**0**	**0**	$\cdots$	**0**	$\cdots$
2	**0**	**1**	**1**	**0**	**0**	$\cdots$	**0**	$\cdots$
3	**0**	**1**	**3**	**1**	**0**	$\cdots$	$\boldsymbol{C(3,k)}$	$\cdots$
4	**0**	**1**	**10**	**6**	**1**	$\cdots$	$\boldsymbol{C(4,k)}$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\boldsymbol{n}$	$\boldsymbol{C(n,0)}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\boldsymbol{C(n,k)}$	$\cdots$

10.5 The type of a permutation and the Cauchy coefficients

Given a permutation σ of a finite n-set, we say that σ has *type*

$$1^{\nu_1}2^{\nu_2}3^{\nu_3}\cdots n^{\nu_n}$$

whenever

$$\sigma \ has\ \nu_i\ cycles\ of\ cardinality\ i,\ for\ i=1,2,\ldots,n.$$

The *Cauchy coefficients* $P(n;1^{\nu_1}2^{\nu_2}\cdots)$ are defined in the following way:

$$C(n;1^{\nu_1}2^{\nu_2}\cdots) \overset{def}{=} \#\ \text{permutations of type } 1^{\nu_1}2^{\nu_2}3^{\nu_3}\cdots \text{ of an n-set.}$$

Clearly,

$$C(n;1^{\nu_1}2^{\nu_2}\cdots) \neq 0$$

if and only if

$$\nu_1+2\nu_2+3\nu_3+\cdots=n.$$

We can compute the Cauchy coefficients by means of the remarkable close form:

Proposition 10.10. *If* $\nu_1 + 2\nu_2 + 3\nu_3 + \cdots = n$, *then*

$$C(n; 1^{\nu_1} 2^{\nu_2} \cdots) = \frac{n!}{1^{\nu_1} 2^{\nu_2} 3^{\nu_3} \ldots} \cdot \frac{1}{\nu_1! \nu_2! \nu_3! \ldots}.$$

Proof. We shall use the (inverse) shepherd's principle. We construct a permutation of type $1^{\nu_1} 2^{\nu_2} \cdots$ by means of the following procedure.

First, we exhibit a partition Π of type $1^{\nu_1} 2^{\nu_2} \cdots$. This can be done in $P(n; 1^{\nu_1} 2^{\nu_2} \cdots)$ ways.

Then, on *any* of the ν_i blocks of cardinality i ($i = 1, 2, 3, \ldots$) of the given partition Π, we construct all the possible cycles: this can be done in $(i-1)!$ ways per block. Therefore, the constructions of cycles on blocks can be performed in a total of

$$((1-1)!)^{\nu_1} ((2-1)!)^{\nu_2} ((3-1)!)^{\nu_3} \cdots$$

different ways.

Thus,

$$C(n; 1^{\nu_1} 2^{\nu_2} \cdots) \ = \ P(n; 1^{\nu_1} 2^{\nu_2} \cdots) \cdot ((1-1)!)^{\nu_1} ((2-1)!)^{\nu_2} ((3-1)!)^{\nu_3} \cdots ,$$

equals (by Proposition 9.5)

$$\frac{n!}{(1!)^{\nu_1} (2!)^{\nu_2} (3!)^{\nu_3} \cdots} \cdot \frac{1}{\nu_1! \nu_2! \nu_3! \ldots} \cdot ((1-1)!)^{\nu_1} ((2-1)!)^{\nu_2} ((3-1)!)^{\nu_3} \cdots$$

which in turn, equals:

$$\frac{n!}{1^{\nu_1} 2^{\nu_2} 3^{\nu_3} \ldots} \cdot \frac{1}{\nu_1! \nu_2! \nu_3! \ldots}.$$

□

10.6 A concluding remark on permutation statistics

From their combinatorial definitions, we immediately infer

Proposition 10.11. *Let* $n \in \mathbb{N}$. *Then*

$$n! \ = \ \sum_{k=0}^{n} C(n,k) \ = \ \sum_{(\nu_1, \nu_2, \ldots)} C(n; 1^{\nu_1} 2^{\nu_2} \cdots).$$

10.7 Derangements

Given a permutation $\sigma : \underline{n} \longleftrightarrow \underline{n}$, a *fixed point* of σ is an element $i \in \underline{n}$ such that

$$\sigma(i) = i.$$

Set

$$Fix(\sigma) = \{i \in \underline{n};\ i \ fixed\ point\ of\ \sigma\}.$$

The permutation $\sigma : \underline{n} \longleftrightarrow \underline{n}$ is said to be a *derangement* whenever it has no fixed points, that is

$$Fix(\sigma) = \emptyset.$$

For positive integers, $n \in \mathbb{Z}^+$, the *derangement numbers* are defined in the following way:

$$d_n \stackrel{def}{=} \# \text{ derangements on an n-set.}$$

Clearly,

$$d_1 = 0, \quad d_2 = 1.$$

Indeed, the unique permutation of $\underline{1}$ is the identity permutation $\sigma(1) = 1$. For $n = 2$, we have two permutations of the set $\underline{2}$:

$$\sigma(1) = 1, \ \sigma(2) = 2, \quad \tau(1) = 2, \ \tau(2) = 1.$$

For $n > 2$, we can compute the derangement numbers d_n by means of the following recursion:

Proposition 10.12. *Given $n \in \mathbb{Z}^+$, with $n > 2$, we have:*

$$d_n = (n-1)(d_{n-2} + d_{n-1}).$$

Proof. We shall use the bad element method. Fix $n \in \underline{n}$ as bad element and consider the length of the (unique) cycle C_n (of a derangement σ) that contains n.

We have two cases:

1. The length of C_n equals 2. In how many ways can we construct these derangements? The 2-cycle C_n can be choosen in $n-1$ different ways; we must only specify the second element of C_n, that is, any $i \in \underline{n-1} = \{1, 2, \ldots, n-1\}$. On the remaining $(n-2)$-set

$$\underline{n-1} - \{i\}$$

we have to construct again a derangement, and this can be done in d_{n-2} ways. Then, the total number in this case will be:

$$(n-1)d_{n-2}.$$

2. The length of C_n is strictly greater than 2. Given such a derangement σ, consider the associated permutation digraph $\vec{G}_\sigma$. This permutation digraph has no loops, and the (unique) cycle C_n that contains n has at least 3 vertices. How can we construct these permutation digraphs? First, we shall costruct a permutation digraph $\vec{G^*}$ (with no loops - derangement!) on the first $n-1$ elements $\underline{n-1}$: this can be done in d_{n-1} different ways. Now, we must insert the bad element n. This can be done putting n on any existing arrow of $\vec{G^*}$, in order to split it into a pair of two consecutive

arrows. But $\vec{G}^*$ is a permutation digraph on $n-1$ vertices. Thus, the number of arrows is $n-1$. Hence, the insertion of the bad element n can be performed in $n-1$ different ways. Therefore, the total number in this second case will be:

$$(n-1)d_{n-1}.$$

Hence,

$$d_n = (n-1)d_{n-2} + (n-1)d_{n-1}.$$

□

Example 10.13. We have:

$$d_1 = 0,\ d_2 = 1,\ d_3 = 2,\ d_4 = 6,\ d_5 = 32,\ d_6 = 190,\ d_7 = 1332,\ d_8 = 10654, \ldots$$

□

11 Some polynomial identities

11.1 The polynomial sequences of power polynomials x^n, rising factorial polynomials $\langle x\rangle_k$ and falling factorial polynomials $(x)_k$

We will consider three sequences of polynomials in the algebra (vector space) $\mathbb{R}[x]$:

– The sequence of *power polynomials*:

$$\{x^n;\ n \in \mathbb{N}\}. \tag{40}$$

– The sequence of *rising factorial polynomials*:

$$\{\langle x\rangle_n;\ n \in \mathbb{N}\}, \tag{41}$$

where

$$\langle x\rangle_0 \stackrel{def}{=} 1, \quad \langle x\rangle_n \stackrel{def}{=} x(x+1)\cdots(x+n-1)\ for\ n > 0.$$

– The sequence of *falling factorial polynomials*:

$$\{(x)_n;\ n \in \mathbb{N}\}, \tag{42}$$

where

$$(x)_0 \stackrel{def}{=} 1, \quad (x)_n \stackrel{def}{=} x(x-1)\cdots(x-n+1)\ for\ n > 0.$$

Cleary, we have

Proposition 11.1. *The sequences* (40), (41), (42) *are* bases *of the vector space of polynomials* $\mathbb{R}[x]$.

11.2 The coefficients $C(n,k)$ and the Stirling numbers of the first kind $s(n,k)$

We have a remarkable expansion formula for the rising factorial polynomials into power polynomials.

Proposition 11.2. *Let $n \in \mathbb{N}$. Then,*

$$\langle x\rangle_n \overset{!}{=} \sum_{k=0}^{n} C(n,k)\ x^k, \tag{43}$$

where the coefficients $C(n,k)$'s are the permutation numbers defined in subsection 10.4.

Proof. We canonically write

$$\langle x\rangle_n \overset{!}{=} \sum_{k=0}^{n} c(n,k)\ x^k, \quad c(n,k) \in \mathbb{R}. \tag{44}$$

Clearly, $c(0,k) = \delta_{0k}$ and $c(n,0) = \delta_{n0}$. Indeed, the polynomials $\langle x\rangle_n$ have zero constant term whenever $n > 0$.

Now, for $n > 0$,

$$\langle x\rangle_n \ = \ \langle x\rangle_{n-1}(x+n-1),$$

that is

$$\langle x\rangle_n \ = \sum_{h=0}^{n-1} c(n-1,h)\ x^{h+1} + (n-1)\left(\sum_{h=0}^{n-1} c(n-1,h)\ x^h\right). \tag{45}$$

By setting (44) = (45) and $h+1 = k$, we get

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k).$$

The double sequences $(\ c(n,k)\ )_{n,k} \in \mathbb{N}$ and $(\ C(n,k)\ )_{n,k} \in \mathbb{N}$ have the same initial conditions and recursion rule: hence, they are equal. □

By definition, the *Stirling numbers of the first kind* $s(n,k)$ are the coefficients in the expansions of falling factorial polynomials into power polynomials, that is

$$(x)_n \overset{def}{=} \sum_{k=0}^{n} s(n,k)\ x^k, \quad s(n,k) \in \mathbb{Z}. \tag{46}$$

By comparing eqs. (43) and (46), it immediately follows

Proposition 11.3. *Let $n,k \in \mathbb{N}$. Then,*

$$s(n,k) = \ (-1)^{n-k}\ C(n,k).$$

11.3 The unique factorization theorem for functions

Let X and Y be sets, $F : X \to Y$ be a function.

Define an equivalence relation $\sim_F$ on the domain set X by setting:

$$x \sim_F x' \Leftrightarrow F(x) = F(x').$$

Let $X/\sim_F$ denote the quotient set of X with respect to $\sim_F$, that is - in the language of partitions - the associated partition $\Pi_{\sim_F}$.

Let

$$\pi : X \stackrel{su}{\to} X/\sim_F, \quad \pi : x \mapsto [x]_{\sim_F}$$

be the *canonical projection.*

Then

Proposition 11.4. *The function F uniquely factorizes into the composition of a surjective function and an injective function*

$$F = \overline{F} \circ \pi,$$

where

$$\overline{F} : X/\sim_F \stackrel{1-1}{\to} Y$$

is the function

$$\overline{F} : [x]_{\sim_F} \mapsto F(x). \tag{47}$$

Proof. First of all, we have to check that definition (47) is well-posed.

But

$$[x]_{\sim_F} = [x']_{\sim_F} \Leftrightarrow x \sim_F x' \Leftrightarrow F(x) = F(x') \Leftrightarrow \overline{F}(\ [x]_{\sim_F}\ ) = \overline{F}(\ [x']_{\sim_F}\ ).$$

The implications from left to right mean that (47) is well-posed. The implications from right to left mean that $\overline{F}$ is injective.

Finally, we have:

$$(\overline{F} \circ \pi)(x) = \overline{F}(\pi(x)) = \overline{F}(\ [x]_{\sim_F}\ ) = F(x).$$

□

We rephrase Proposition 11.4 in the following way:

Corollary 11.5. *Let Π be a partition of the set X. The map*

$$F \ \mapsto \ \overline{F}$$

defines a bijection

$$\{\ F : X \to Y;\ X/\sim_F = \Pi\ \} \ \longleftrightarrow \ \{\ \overline{F} : \Pi \stackrel{1-1}{\to} Y\ \}.$$

Now, assume that both X and Y are finite, say $X = \underline{n}$, $Y = \underline{m}$. Then,

$$m^n = |\{F : \underline{n} \to \underline{m}\}| = | \dot{\bigcup}_{\Pi \ partition \ of \ \underline{n}} (\{ \ F : \underline{n} \to \underline{m}; \ \underline{n}/ \sim_F = \Pi \ \}) \, |$$

equals, by Corollary 11.5,

$$| \dot{\bigcup}_{\Pi \ partition \ of \ \underline{n}} \left(\{ \ \overline{F} : \Pi \ \stackrel{1-1}{\to} \ \underline{m} \ \}\right) | = \sum_{\Pi \ partition \ of \ \underline{n}} |\{ \ \overline{F} : \Pi \ \stackrel{1-1}{\to} \ \underline{m} \ \}|$$

which in turn, equals

$$\sum_{k=0}^{n} \left(\sum_{\Pi \ k-partition \ of \ \underline{n}} |\{ \ \overline{F} : \Pi \ \stackrel{1-1}{\to} \ \underline{m} \ \}| \right) = \sum_{k=0}^{n} S(n,k) \ (m)_k.$$

Therefore, we proved the following class of combinatorial identities:

Corollary 11.6.

Let $m, n \in \mathbb{N}$. Then,

$$m^n \ = \sum_{k=0}^{n} S(n,k) \ (m)_k.$$

11.4 The Stirling numbers of the 2nd kind $S(n,k)$ as expansion coefficients in the vector space of polynomials $\mathbb{R}[x]$

Recall that, given a non zero polynomial of degree $deg(p(x)) = n$, $n \in \mathbb{N}$, say

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in \mathbb{R}[x], \quad a_n \neq 0$$

and a real number $\alpha \in \mathbb{R}$, the *evaluation of* $p(x)$ *at* α is the real number

$$E_\alpha(p(x)) = a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n \in \mathbb{R}.$$

A real number α is said to be a *root* of a non zero polynomial $p(x)$ whenever

$$E_\alpha(p(x)) = a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n = 0.$$

From *Ruffini's Theorem*, it follows:

Proposition 11.7. *The number of roots (even counted with multiplicities) of a non zero polynomial* $p(x)$ *is less than or equal to its degree* $deg(p(x))$.

From this, it follows:

Corollary 11.8. *(The identity principle for polynomials) Let $p(x), q(x) \in \mathbb{R}[x]$ be non non zero polynomials and assume that they admit an* infinite *family of equal evaluations, say*

$$E_{\alpha_m}(p(x)) = E_{\alpha_m}(q(x)), \quad \alpha_m \in \mathbb{R}, \quad m \in \mathbb{N}.$$

Then,

$$p(x) = q(x).$$

Proof. Clearly,

$$E_{\alpha_m}(p(x)) = E_{\alpha_m}(q(x)) \iff \alpha_m \textit{ is a root of } p(x) - q(x)$$

for $m \in \mathbb{N}$. From Proposition 11.7, it follows

$$p(x) - q(x) \equiv 0 \iff p(x) = q(x).$$

□

Notice that we are now able to rewrite Corollary 11.3 as follows:

Corollary 11.9. *Let $n \in \mathbb{N}$. Then,*

$$E_m(x^n) \ = \ m^n \ = \sum_{k=0}^{n} S(n,k) \ (m)_k \ = \ E_m(\sum_{k=0}^{n} S(n,k) \ (x)_k \ ),$$

for every $m \in \mathbb{N}$.

Therefore, from Corollary 11.8 we infer the remarkable polynomial identities in $\mathbb{R}[x]$:

Proposition 11.10. *Let $n \in \mathbb{N}$. Then,*

$$x^n \ = \sum_{k=0}^{n} S(n,k) \ (x)_k \in \mathbb{R}[x]. \tag{48}$$

11.5 The Theorem $\mathbf{s} \ = \ \mathbf{S}^{-1}$

Recall that, for the Stirling numbers of the first kind, we have (see eq. (46))

$$(x)_n \stackrel{def}{=} \sum_{k=0}^{n} s(n,k) \ x^k, \tag{49}$$

and, for the Stirling numbers of the second kind (see eq. (48))

$$x^n \ = \sum_{k=0}^{n} S(n,k) \ (x)_k. \tag{50}$$

In the vector space $\mathbb{R}[x]$:

- Relation (49) means that the matrix

$$\mathbf{s} = [\ s(n,k)\ ]_{n,k} \in \mathbb{N}$$

is the transition matrix from the basis of *falling factorial polynomials*

$$\{(x)_n;\ n \in \mathbb{N}\}.$$

to the basis of *power polynomials*

$$\{x^n;\ n \in \mathbb{N}\}.$$

- Relation (50) means that the matrix

$$\mathbf{S} = [\ S(n,k)\ ]_{n,k} \in \mathbb{N}$$

is the transition matrix from the basis of *power polynomials*

$$\{x^n;\ n \in \mathbb{N}\}$$

to the basis of the *falling factorial polynomials*

$$\{(x)_n;\ n \in \mathbb{N}\}.$$

From *Linear Algebra*, it follows:

Proposition 11.11. *We have*

$$\mathbf{s} \times \mathbf{S} = \mathbf{Id} = \mathbf{S} \times \mathbf{s},$$

that is,

$$\mathbf{s}^{-1} = \mathbf{S} \iff \mathbf{s} = \mathbf{S}^{-1}.$$

12 An introduction to the Moebius-Rota inversion theory

12.1 A glimpse on the general case

The Rota theory of Moebius inversion is a fairly general one, as it applies to all *locally finite partially ordered sets.*

The foundational contribution is Rota's paper [4] of 1964. For a comprehensive treatment of the theory, we refer the reader to [5].

A *partially ordered set* (*poset*, for short) is a pair $(P, \leq)$ where P is a set and $\leq$ is a *partial order relation.* This, in turn, is a binary relation $R \subseteq P \times P$ such that

1. $(x, x) \in R$ (reflexivity)

2. $(x,y) \in R \Rightarrow (y,x) \notin R$ if $x \neq y$: (antisymmetry)

3. $(x,y),\ (y,z) \in R \Rightarrow (x,z) \in R$ (transitivity)

Clearly, when we consider an order relation, we simply write $x \leq y$ for xRy.

For the sake of simplicity, we will assume in the following that $(P, \leq)$ is a *finite* poset, that is $|P| < \infty$.

Let $\mu_P : P \times P \to \mathbb{Z}$ be the *unique* function that satisfies the following conditions:

1. $\mu_P(x,y) = 0$ if $x \not\leq y$,

2. $\mu_P(x,x) = 1$, $\forall\ x \in P$,

3. $\mu_P(x,y) = -\sum_{z:x\leq z<y}\ \mu_P(x,z) = -\sum_{z:x<z\leq y}\ \mu_P(z,y)$ if $x < y$.

The function μ_P is the *Moebius function* of the poset P. When no confusion might arise, we will simply write μ in place of μ_P.

Define the auxiliary functions

$$\zeta, \delta : P \times P \to \mathbb{Z}$$

in the following way:

$$\zeta(x,y) = 1 \textit{ whenever } x \leq y, \quad \zeta(x,y) = 0 \textit{ otherwise},$$

$$\delta(x,y) = 1 \textit{ whenever } x = y, \quad \delta(x,y) = 0 \textit{ otherwise}.$$

The next result immediately follows from the definitions:

Proposition 12.1. *For every $z \in P$, we have*

$$\sum_{x \leq y} \zeta(z,x)\ \mu(x,y) \ = \ \delta(z,y)$$

$\square$

The following result, simple though it is, is fundamental.

Theorem 12.2. (Moebius inversion formula) *Let*

$$f, g : P \longrightarrow \mathbb{R}$$

be real-valued functions such that

$$\sum_{x \leq y} f(x) \ = \ g(y), \quad \forall y \in P. \tag{51}$$

Then

$$f(y) \ = \ \sum_{x \leq y} \mu(x,y)\ g(x), \quad \forall y \in P. \tag{52}$$

Proof. Substituting the right side of (51) into the right side of (52) and simplifying,

$$\sum_{x \leq y} \mu(x,y)\ g(x) \ = \ \sum_{x \leq y} \sum_{z \leq x} f(z)\ \mu(x,y). \tag{53}$$

The right side of (53) is then rewritten in the form

$$\sum_{x \leq y} \sum_{z} \zeta(z,x)\ f(z)\ \mu(x,y).$$

Interchanging the order of summation, this becomes

$$\sum_{z} f(z) \sum_{x \leq y} \zeta(z,x)\ \mu(x,y) \ = \ \sum_{z} f(z)\ \delta(z,y) \ = \ f(y),$$

by Proposition 12.1.

□

Therefore, the crucial problem of the theory is to determine *explicit/close form* formulae for the Moebius functions of different classes of posets.

We limit ourselves to recall two classical (and fundamental) cases.

Example 12.3. (The set-theoretic case) Let S be a finite set. Let $(P, \leq) = (\mathbb{P}(S), \subseteq)$, that is, P is the power set $\mathbb{P}(S) = \{A;\ A \subseteq S\}$ and the order is the inclusion $\subseteq$.

Proposition 12.4. *Let μ be the Moebius function of $(\mathbb{P}(S), \subseteq)$. Then*

$$\mu(A,B) \ = \ (-1)^{|B|-|A|}\ if\ A \subseteq B, \quad \mu(A,B) = 0\ if\ A \not\subseteq B.$$

Proof. We have to prove that, given $A \subseteq B \subseteq S$, $A \neq B$, we have:

$$\sum_{C : A \subseteq C \subseteq B} (-1)^{|B|-|C|} = 0. \tag{54}$$

Set $|A| = k$, $|B| = m$, $k < m$. Eq. (54) can be rewritten as

$$\begin{aligned} \sum_{h=k}^{m} \sum_{C : A \subseteq C \subseteq B,\ |C| = h} (-1)^{m-h} &= \sum_{h=k}^{m-k} \binom{m-k}{h-k} (-1)^{m-h} \\ &= \sum_{j=0}^{m-k} \binom{m-k}{j} (-1)^{m-k-j} = 0. \end{aligned}$$

□

Example 12.5. (The classical Moebius function of Number Theory) Let $(P, \leq) = (\mathbb{Z}^+, |\ )$ be the poset of positive integers, endowed with the partial order relation *divide*, that is:

$$m \mid n \overset{def}{\Leftrightarrow} n = hm,\ h \in \mathbb{Z}^+.$$

The poset $(\mathbb{Z}^+, |\ )$ is not finite, but it is a *locally finite* poset, with *minimum*, the positive integer 1.

Proposition 12.6. *Let μ be the Moebius function of the poset $(\mathbb{Z}^+, |\ )$ Then,*

1. $\mu(1, n) = 1$ *if n is a square-free positive integer with an* even *number of prime factors.*
2. $\mu(1, n) = -1$ *if n is a square-free positive integer with an* odd *number of prime factors.*
3. $\mu(1, n) = 0$ *if n has a squared prime factor.*
4. $\mu(m, n) = \mu(1, \frac{n}{m}) \quad$ *if* $m|n$.
5. $\mu(m, n) = 0 \quad$ *if* m *doesn't divide* n.

$\square$

12.2 The Moebius inversion principle (set-theoretic case)

We explicitly restate Theorem 12.2 for the posets (boolean algebras)

$$(\mathbb{P}(S), \subseteq),$$

where S is a finite set.

Proposition 12.7. (Set-theoretic Moebius inversion formula) *Let S be a finite set. Let*

$$f, g : \mathbb{P}(S) \longrightarrow \mathbb{R}$$

be real-valued functions such that

$$\sum_{A \subseteq B} f(A) \ = \ g(B), \quad \forall\ B \subseteq S. \tag{55}$$

Then,

$$f(B) \ = \ \sum_{A \subseteq B} (-1)^{|B|-|A|}\ g(A), \quad \forall\ B \subseteq S. \tag{56}$$

12.2.1 On the number of surjective functions

Our problem is to discover and prove a *close form* formula for the numbers

$$\#\{F : \underline{k} \overset{su}{\to} \underline{n}\}.$$

We proceed by Moebius inversion on the power set $(\mathbb{P}(\underline{n}), \subseteq)$.
Define

$$f : \mathbb{P}(\underline{n}) \ \to \ \mathbb{R}$$

by setting

$$f(A) \overset{def}{=} \#\{F : \underline{k} \to \underline{n};\ Im(F) = A\}, \quad \forall\ A \subseteq \underline{n}.$$

Define

$$g : \mathbb{P}(\underline{n}) \ \to \ \mathbb{R}$$

by setting

$$g(B) \stackrel{def}{=} \# \{F : \underline{k} \to \underline{n};\ Im(F) \subseteq B\}, \quad \forall\ B \subseteq \underline{n}.$$

Since

$$\{F : \underline{k} \to \underline{n};\ Im(F) \subseteq B\} \ = \dot{\bigcup}_{A \subseteq B} \ \{F : \underline{k} \to \underline{n};\ Im(F) = A\} \quad \forall\ B \subseteq \underline{n},$$

then,

$$\sum_{A \subseteq B} f(A) \ = \ g(B), \quad \forall\ B \subseteq \underline{n}.$$

By Moebius inversion (56) and by setting $|B| = m$, we infer that

$$\begin{aligned} f(B) \ &= \ \sum_{A \subseteq B} (-1)^{|B|-|A|}\ g(A) \quad \forall\ B \subseteq \underline{n} \\ &= \ \sum_{j=0}^{m} (-1)^{m-j} \sum_{|A|=j, A \subseteq B} m^j \\ &= \ \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j}\ j^k. \end{aligned}$$

By specializing to $B = \underline{n}$, we get:

Proposition 12.8. *We have:*

$$\#\{F : \underline{k} \stackrel{su}{\to} \underline{n}\} \ = \ \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j}\ j^k.$$

12.3 The dual Moebius inversion principle (set-theoretic case)

We explicitly restate Proposition 12.7 in *dual form*, that is, for the posets (boolean algebras)

$$(\mathbb{P}(S), \supseteq\ ),$$

where $\supseteq$ denotes the *reverse inclusion order*.

Proposition 12.9. (Dual set-theoretic Moebius inversion formula) *Let S be a finite set. Let*

$$f, g : \mathbb{P}(S) \longrightarrow \mathbb{R}$$

be real-valued functions such that

$$\sum_{A \supseteq B} f(A) \ = \ g(B), \quad \forall\ B \subseteq S. \tag{57}$$

Then,

$$f(B) \ = \ \sum_{A \supseteq B} (-1)^{|A|-|B|}\ g(A), \quad \forall\ B \subseteq S. \tag{58}$$

12.3.1 The generalized derangement problem

Let $n, k \in \mathbb{N}$ and consider the numbers:

$$d_{n,k} \overset{def}{=} \# \text{ n-permutations with } \underline{\text{exactly}} \text{ k fixed points.}$$

Clearly, $d_{n,0} = d_n$, the number of derangements on n points.

Proposition 12.10. *We have:*

$$d_{n,k} \ = \ \binom{n}{k} \, d_{n-k}.$$

Proof. We choose the k-subsets of fixed points in $\binom{n}{k}$ different ways. Then, we multiply by the number d_{n-k} of derangements on the remaining $n-k$ points. $\square$

Our problem is to discover and prove a *close form* formula for the numbers

$$d_{n,k}.$$

We proceed by Moebius inversion on $(\mathbb{P}(\underline{n}), \supseteq)$.
Fix a subset $B \subseteq \underline{n}$, with $|B| = k$. Clearly,

$$\{\sigma : \underline{n} \leftrightarrow \underline{n};\ Fix(\sigma) \supseteq B\} \ = \dot{\bigcup}_{A \supseteq B} \ \{\sigma : \underline{n} \leftrightarrow \underline{n};\ Fix(\sigma) = A\}.$$

Then, if we set

$$f : \mathbb{P}(\underline{n}) \ \to \ \mathbb{R},$$

$$f(A) \overset{def}{=} \# \{\sigma : \underline{n} \leftrightarrow \underline{n};\ Fix(\sigma) = A\}, \quad \forall \ A \subseteq \underline{n}.$$

and

$$g : \mathbb{P}(\underline{n}) \ \to \ \mathbb{R},$$

$$g(B) \overset{def}{=} \# \{\sigma : \underline{n} \leftrightarrow \underline{n};\ Fix(\sigma) \supseteq B\}, \quad \forall \ A \subseteq \underline{n},$$

we get

$$\sum_{A \supseteq B} f(A) \ = \ g(B), \quad \forall \ B \subseteq \underline{n}.$$

By dual Moebius inversion (12.9), we infer

$$\begin{aligned}
f(B) \ &= \ \sum_{A \supseteq B} (-1)^{|A|-|B|} \ g(A) \\
&= \ \sum_{h=k}^{n} (-1)^{h-k} \sum_{|A|=h, A \supseteq B} g(A) \\
&= \ \sum_{h=k}^{n} (-1)^{h-k} \binom{n-k}{h-k} \ (n-h)! \\
&= \ \sum_{h=k}^{n} (-1)^{h-k} \frac{(n-k)!}{(n-h)!(h-k)!} \ (n-h)! \\
&= \ \sum_{h=k}^{n} (-1)^{h-k} \frac{(n-k)!}{(h-k)!}.
\end{aligned}$$

Proposition 12.11. *We have*

$$d_{n,k} \;=\; \frac{n!}{k!} \sum_{h=k}^{n} \frac{(-1)^{h-k}}{(h-k)!}.$$

Proof. The choices of the subset $B \subseteq \underline{n}$, $|B| = k$ are $\binom{n}{k}$. Then,

$$\begin{aligned} d_{n,k} &= \binom{n}{k} f(B) \\ &= \binom{n}{k} \sum_{h=k}^{n} (-1)^{h-k} \frac{(n-k)!}{(h-k)!} \\ &= \frac{n!}{k!(n-k)!} \sum_{h=k}^{n} (-1)^{h-k} \frac{(n-k)!}{(h-k)!} \\ &= \frac{n!}{k!} \sum_{h=k}^{n} \frac{(-1)^{h-k}}{(h-k)!}. \end{aligned}$$

□

In the case of *derangements*, that is, $k = 0$, we get:

Corollary 12.12. *We have:*

$$d_n \;= d_{n,0} \;=\; n! \sum_{h=0}^{n} \frac{(-1)^h}{h!}.$$

We provide a beautiful *probabilistic* version/interpretation of this fact.

Clearly, the *probability* $\mathbf{P}_n$ that an n-permutation is a *derangement* is given by

$$\mathbf{P}_n \;=\; \frac{d_n}{n!} \;=\; \sum_{h=0}^{n} \frac{(-1)^h}{h!}.$$

Then, the *asymptotic* $\mathbf{P}_n \overset{n\to\infty}{\longrightarrow} \mathbf{P}_\infty$ of this probability is:

$$\mathbf{P}_\infty \;=\; \sum_{h=0}^{\infty} \frac{(-1)^h}{h!} \;=\; \frac{1}{\mathbf{e}},$$

where $\mathbf{e}$ denotes the *Nepero number* (the basis of natural logarithms).

As a matter of fact,

$$\sum_{h=0}^{\infty} \frac{(-1)^h}{h!}$$

is the evaluation at -1 of the Taylor expansion of the exponential

$$\mathbf{e}^x \;=\; \sum_{h=0}^{\infty} \frac{x^h}{h!}, \quad x \in \mathbb{R}.$$

12.4 Sieve method

Let $\mathbf{\Omega}$ be a finite set (sample space) and let $A_1, A_2, \ldots, A_n$ be subsets $A_i \subseteq \mathbf{\Omega}$ (forbidden events). Let $\underline{n} = \{1, 2, \ldots, n\}$ be the *family of indexes*.

12.4.1 Complete products

Given $T \subseteq \underline{n}$, consider the subset

$$\left(\bigcap_{i\in T} A_i\right) \bigcap \left(\bigcap_{i\notin T} A_i^c\right) \subseteq \mathbf{\Omega}, \tag{59}$$

where A_i^c denotes the complementary set $\mathbf{\Omega} \smallsetminus A_i$ of A_i in $\mathbf{\Omega}$.

The subset (59) is called the *complete product* associated to the subfamily of indexes $T \subseteq \underline{n}$.

The subset (59) is the set

$$\{x \in \mathbf{\Omega};\ x \in A_i\ for\ i \in T,\ x \notin A_i\ for\ i \notin T\}. \tag{60}$$

Complete products are pairwise *disjoint*. Furthermore, from (60) one easily recognizes:

Proposition 12.13. *Given $S \subseteq \underline{n}$, we have*

$$\bigcap_{i\in S} A_i \ = \dot{\bigcup}_{T\supseteq S} \left(\bigcap_{i\in T} A_i\right) \bigcap \left(\bigcap_{i\notin T} A_i^c\right).$$

Define two functions
$$f, g : \mathbb{P}(\underline{n}) \to \mathbb{R}$$
in the following way

$$f(T) \ = \ \left|\left(\bigcap_{i\in T} A_i\right) \bigcap \left(\bigcap_{i\notin T} A_i^c\right)\right|,\ \forall\ T \subseteq \underline{n},$$

$$g(S) \ = \ \left|\bigcap_{i\in S} A_i\right|,\ \forall\ S \subseteq \underline{n}.$$

Proposition 12.13 reads as:

$$\sum_{T\supseteq S} f(T) \ = \ g(S),\ \forall\ S \subseteq \underline{n}.$$

By dual Moebius inversion on $\mathbb{P}(\underline{n})$, we obtain

Proposition 12.14. *Given $S \subseteq \underline{n}$, $|S| = m$, we have:*

$$\begin{aligned} f(S) &= \left| \left(\bigcap_{i \in S} A_i \right) \bigcap \left(\bigcap_{i \notin S} A_i^c \right) \right| \\ &= \sum_{T \supseteq S} (-1)^{|T|-|S|} \; g(T), \; \forall \; S \subseteq \underline{n} \\ &= \sum_{k=m}^{n} (-1)^{k-m} \sum_{|T|=k, T \supseteq S} g(T) \\ &= \sum_{k=m}^{n} (-1)^{k-m} \sum_{|T|=k, T \supseteq S} \left| \bigcap_{i \in T} A_i \right|. \end{aligned}$$

12.4.2 The formula of Sylvester

The problem: Let $\mathbf{\Omega}$ be a finite set (sample space) and let $A_1, A_2, \ldots, A_n$ be subsets $A_i \subseteq \mathbf{\Omega}$ (forbidden events). Let $\underline{n} = \{1, 2, \ldots, n\}$ be the *family of indexes*. Compute the cardinality (more in general: probability/measure)

$$\left| \mathbf{\Omega} \smallsetminus \bigcup_{i=1}^{n} A_i \right|$$

by an *efficient close form* formula.

Consider the natural integers:

$$\mathbf{S}_k \stackrel{def}{=} \sum_{|T|=k, T \subseteq \underline{n}} \left| \bigcap_{i \in T} A_i \right|.$$

These numers are called *Sylvester numbers.*

The cardinality

$$\left| \mathbf{\Omega} \smallsetminus \bigcup_{i=1}^{n} A_i \right|$$

can be computed by means of an *alternating signs* sum of the Sylvester numbers.

Proposition 12.15. (The formula of Sylvester) *We have*

$$\left| \mathbf{\Omega} \smallsetminus \bigcup_{i=1}^{n} A_i \right| = \sum_{k=0}^{n} (-1)^k \; \mathbf{S}_k.$$

Proof. From Proposition 12.14. Specialize to $S = \emptyset$: then,

$$f(\emptyset) = \left| \bigcap_{i=1}^{n} (A_i)^c \right| = \left| \mathbf{\Omega} \smallsetminus \bigcup_{i=1}^{n} A_i \right|,$$

by the DeMorgan laws (elementary, from high school math).

Then,

$$\begin{aligned}
\left| \mathbf{\Omega} \setminus \bigcup_{i=1}^{n} A_i \right| &= f(\emptyset) \\
&= \sum_{T \subseteq \underline{n}} (-1)^{|T|} \; g(T) \\
&= \sum_{k=0}^{n} (-1)^k \sum_{|T|=k,\; T \subseteq \underline{n}} \left| \bigcap_{i \in T} A_i \right| \\
&= \sum_{k=0}^{n} (-1)^k \; \mathbf{S}_k .
\end{aligned}$$

□

Example 12.16. *Sieve* originates from *inclusion/exclusion.*

For $n = 3$, Proposition 12.15 reads:

$$\begin{aligned}
|\; \mathbf{\Omega} \setminus \cup_{i=1}^{3} A_i | = \; & |\mathbf{\Omega}| - |A_1| - |A_2| - |A_3| \\
& + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|.
\end{aligned}$$

□

12.4.3 Ch. Jordan's formula

Let $\mathbf{\Omega}$ be a finite set (sample space) and let $A_1, A_2, \ldots, A_n$ be subsets $A_i \subseteq \mathbf{\Omega}$ (forbidden events). Let $\underline{n} = \{1, 2, \ldots, n\}$ be the *family of indexes.*

Given $m = 0, 1, \ldots, n$, let

$$\mathbf{e}_m$$

be the number of elements of $\mathbf{\Omega}$ that belongs to *exactly* m of the forbidden events $A_1, A_2, \ldots, A_n$.

In the notation of Proposition 12.14, we have:

$$\begin{aligned}
\mathbf{e}_m &= \sum_{S\subseteq \underline{n},\ |S|=m} f(S) \\
&= \sum_{S\subseteq \underline{n},\ |S|=m} \sum_{T\supseteq S} (-1)^{|T|-|S|}\ g(T) \\
&= \sum_{k=m}^{n} \sum_{T\subseteq \underline{n},\ |T|=k} \sum_{S\subseteq T, |S|=m} (-1)^{|T|-|S|}\ g(T) \\
&= \sum_{k=m}^{n} (-1)^{k-m} \binom{k}{m} \sum_{|T|=k,\ T\subseteq \underline{n}} g(T) \\
&= \sum_{k=m}^{n} (-1)^{k-m} \binom{k}{m} \sum_{|T|=k,\ T\subseteq \underline{n}} \left| \bigcap_{i\in T} A_i \right| \\
&= \sum_{k=m}^{n} (-1)^{k-m} \binom{k}{m} \mathbf{S}_k.
\end{aligned}$$

Hence,

Proposition 12.17. (Ch. Jordan's formula) *Given $m = 0, 1, \ldots, n$, we have*

$$\mathbf{e}_m = \sum_{k=m}^{n} (-1)^{k-m} \binom{k}{m} \mathbf{S}_k.$$

Clearly, for $m = 0$, we find again the formula of Sylvester.

The approach to the *generalized derangement problem* is even simpler and intuitive, via Ch. Jordan's formula.

Let $\mathbf{\Omega}$ be the set of n-permutations:

$$\mathbf{\Omega} = \{\sigma : \underline{n} \longleftrightarrow \underline{n}\}.$$

and consider the n-family of forbidden events

$$A_i = \{\sigma : \underline{n} \longleftrightarrow \underline{n};\ \sigma(i) = i\}, \quad i = 1, 2, \ldots, n.$$

Then,

$$d_{n,k} = \mathbf{e}_k.$$

From Ch. Jordan's formula, by setting $m = k$ and $k = h$, we get

$$\begin{aligned}
d_{n,k} = \mathbf{e}_k &= \sum_{h=k}^{n} (-1)^{h-k} \binom{h}{k} \binom{n}{h} (n-h)! \\
&= \frac{n!}{k!} \sum_{h=k}^{n} \frac{(-1)^{h-k}}{(h-k)!}.
\end{aligned}$$

12.5 The problem of menages

The *menage problem* asks for the number of different ways in which it is possible to seat a set of male-female couples at a round dining table so that women and men alternate and nobody sits next to his or her partner. This problem was formulated in 1891 by Édouard Lucas and, independently, a few years earlier, by Peter Guthrie Tait in connection with knot theory.

The problem was solved by J. Touchard in 1934. Touchard's approach was simplified by I. Kaplansky in 1943.

12.5.1 A preliminary remark on the "circular" Gergonne problem

Problem. In how many ways can we choose a k-subset S of seats in a round table with seats labelled $\underline{2n} = \{1, 2, 3, \ldots, 2n-1, 2n\}$ such that no two seats in S are *adjiacent*?

Solution. We have to apply two times the solution of the classical Gergonne problem twice.

Consider the following cases.

i) Suppose that the k-subset S contains the seat 1. Then, we have to choose $k-1$ seats in the set $\{3, 4, \ldots, 2n-1\}$, keeping in mind that no two seats must be adjacent. Equation 30, implies that it can be done in

$$\binom{2n-3-k+1+1}{k-1} = \binom{2n-k-1}{k-1}$$

ways.

ii) Suppose that the k-subset S doesn't contain the seat 1. Then, we have to choose k seats in the $2n-1$-set $\{2, 3, \ldots, 2n\}$, keeping in mind that no two seats must be adjacent. Equation 30, implies that it can be done in

$$\binom{2n-1-k+1+1}{k} = \binom{2n-k}{k}$$

ways.

Then, the solution is provided by the number

$$\binom{2n-k-1}{k-1} + \binom{2n-k}{k} = \frac{2n}{2n-k}\binom{2n-k}{k}. \tag{61}$$

12.5.2 The formulae of Touchard and Kaplansky

To begin with, we consider the *reduced* problem of menages: the n women $\underline{n} = \{1, 2, \ldots, n\}$ are already sitting in the *odd* seats $\{1, 3, \ldots, 2n-1\}$.

We must place the n men, say $\underline{n'} = \{1', 2', \ldots, n'\}$.

Their placements are described by the function

$$f : \underline{n} = \{1, 2, \ldots, n\} \longrightarrow \underline{n'} = \{1', 2', \ldots, n'\}$$

such that

$$f(i) \stackrel{def}{=} \text{ the man f(i) sitted on the right of his partner i.} \tag{62}$$

Our aim is to apply the formula of Sylvester. The set $\mathbf{\Omega}$ is

$$\mathbf{\Omega} \;=\; \{\; f : \underline{n} = \{1, 2, \ldots, n\} \longrightarrow \underline{n'}\} = \{1', 2', \ldots, n'\}.$$

In order for the *placement function* f obey to the restrictions of the *menage problem*, we must consider the following $2n$ <u>*forbidden events*</u> $A_i \subseteq \mathbf{\Omega}$, $i = 1, 2, \ldots, 2n$:

1. For $i = 1, 2, \ldots, n$, let

$$A_{2n-1} \;=\; \{\; f : \underline{n} \to \underline{n'},\; f(i) = i'\},$$

2. For $i = 1, 2, \ldots, n-1$, let

$$A_{2i} \;=\; \{\; f : \underline{n} \to \underline{n'},\; f(i) = (i+1)'\},$$

3. Let

$$A_{2n} \;=\; \{\; f : \underline{n} \to \underline{n'},\; f(n) = 1'\}.$$

Therefore, the solution of the reduced problem is given by the positive integer

$$\left| \mathbf{\Omega} \;\setminus\; \bigcup_{j=1}^{2n} A_j \right|. \tag{63}$$

From the Sylvester formula, it follows that (63) equals:

$$\sum_{k=0}^{2n} (-1)^k \sum_{T \subseteq \underline{2n}:\; |T|=k} \left| \bigcap_{j \in T} A_j \right|. \tag{64}$$

But now

$$\bigcap_{j \in T} A_j \;=\; \emptyset,$$

whenever T contains adjacent elements, and

$$\left| \bigcap_{j \in T} A_j \right| \;=\; (n-k)!,$$

otherwise.

Hence, by applying formula (61), the solution of the reduced problem is given by the *Touchard numbers*:

$$\mathbf{U_n} \;=\; \sum_{k=0}^{2n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)! \tag{65}$$

Now, we have to pass from the reduced problem to the general one. To wit:

i) women may be sitting in the odd seats into *any* order: then, there are $n!$ different cases.

ii) women may be sitting either in the odd seats or in the even seats: then, there are 2 different cases.

Hence, the solution of the general menage problem is:

$$2\ n!\ \mathbf{U_n} \ = \ \sum_{k=0}^{2n} (-1)^k \ \frac{4n}{2n-k} \binom{2n-k}{k} \ n!(n-k)! \tag{66}$$

12.6 The Euler Φ function

The *Euler Φ function* is the function

$$\Phi : \mathbb{Z}^+ \to \mathbb{Z}^+$$

such that

$$\Phi(n) \stackrel{def}{=} \ |\{m \in \underline{n};\ GCD(m,n) = 1\}|, \quad n \in \mathbb{Z}^+,$$

that is the number of positive integers m less than or equal to n that are *coprime* with n.

Proposition 12.18. (Euler's Theorem) *Let $n \in \mathbb{Z}^+$, and let*

$$n \ \stackrel{!}{=} \ p_1^{i_1} p_2^{i_2} \cdots p_r^{i_r}$$

($p_1, p_2, \ldots p_r$ pairwise distinct primes*, different from 1) be its* unique factorization *into product of powers of primes. Then*

$$\Phi(n) \ = \ n \ \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_r}\right).$$

Proof. It is another beautiful application of the formula of Sylvester.

Set $\mathbf{\Omega} \ = \ \underline{n}$ and

$$A_i \ = \ \{m \in \underline{n};\ p_i \mid n\}, \quad i = 1, 2, \ldots, r.$$

Then,

$$\Phi(n) \ = \ |\mathbf{\Omega} - \cup_{i=1}^{r} \ A_i|.$$

Now,

$$|A_i| \ = \ \frac{n}{p_i}, \quad i = 1, 2, \ldots, r.$$

In general, given $T \subseteq \underline{r}$, $|T| = k$, say

$$T \ = \ \{i_1, i_2, \ldots, i_k\},$$

then,

$$\left| \bigcap_{i \in T} A_i \right| \ = \ \frac{n}{p_{i_1} p_{i_2} \cdots p_{i_k}}.$$

Hence,

$$\begin{aligned}
\Phi(n) &= n(1 - \frac{1}{p_1} - \cdots - \frac{1}{p_r} + \sum_{p,q} \frac{1}{p_{i_p} p_{i_q}} \\
&\quad - \sum_{p,q,s} \frac{1}{p_{i_p} p_{i_q} p_{i_s}} + \cdots + (-1)^r \frac{1}{p_1 p_2 \cdots p_r} \\
&= n \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_r}\right).
\end{aligned}$$

$\square$

Example 12.19. We have:

1. $n = 30 = 2 \cdot 3 \cdot 5$. Then, $\Phi(30) \; = \; 30(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5}) = 8$.
2. $n = 100 = 2^2 \cdot 5^2$. Then, $\Phi(100) \; = \; 100(1 - \frac{1}{2})(1 - \frac{1}{5}) = 40$.
3. $n = 125 = 5^3$. Then, $\Phi(125) \; = \; 125(1 - \frac{1}{5}) = 100$.
4. $n = 210 = 2 \cdot 3 \cdot 5 \cdot 7$. Then, $\Phi(210) \; = \; 210(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5})(1 - \frac{1}{7}) = 48$.

$\square$

12.7 A glimpse on RSA public-key cryptography

12.7.1 Preliminaries. Congruences <u>*mod n*</u> on the integers $\mathbb{Z}$

Given an integer $n \in \mathbb{Z}$, $n > 1$, we define an equivalence relation $\equiv (mod\ n)$ in the following way.

Given $x, y \in \mathbb{Z}$,

$$x \equiv y(mod\ n) \overset{def}{\iff} x - y \; = \; qn, \quad q \in \mathbb{Z}.$$

Therefore, $x \equiv y(mod\ n)$ if and only if x and y have the same *remainder* with respect to the division by n, i.e.:

$$x \overset{!}{=} q_1 n + r_1, \; y \overset{!}{=} q_2 n + r_2, \; 0 \leq r_1, r_2 < n \; \Rightarrow \; r_1 \; = \; r_2.$$

Hence, the equivalence classes with respect to $\equiv (mod\ n)$ are canonically represented by the <u>remainders</u> with respect to the division by n:

$$[0]_{mod\ n}, \; [1]_{mod\ n}, \ldots, \; [n-1]_{mod\ n}.$$

The equivalence relations $= (mod\ n)$ are *congruences*, that is, are compatible with the operations of addition and sum.

If $x \equiv y(mod\ n)$ and $x' \equiv y'(mod\ n)$, then

$$x + x' \; \equiv \; y + y'(mod\ n) \quad and \quad xx' \; \equiv \; yy'(mod\ n).$$

We limit ourselves to recall a couple of elementary facts.

Proposition 12.20. *Let $n \in \mathbb{Z}^+$, $n > 1$, and let $x \in \mathbb{Z}^+$, $0 < x < n$, such that $GCD(x, n) = 1$. There exists a* unique *$y \in \mathbb{Z}^+$, $0 < y < n$, such that*

$$xy \equiv 1(mod\ n). \tag{67}$$

□

Remark 12.21. *Under condition (12.20), we write (by consistent convention):*

$$y \equiv x^{-1}(mod\ n), \quad [y]_{mod\ n} = ([x]_{mod\ n})^{-1}.$$

□

Example 12.22. For example, let $n = 20$, $x = 3$, $y = 7$. Since

$$xy = 21 \equiv 1(mod\ 20),$$

then,

$$7 \equiv 3^{-1}(mod\ 20),$$

and

$$[7]_{mod\ 20} = ([3]_{mod\ 20})^{-1}.$$

Furthermore,

$$7 \equiv -13(mod\ 20),$$

then,

$$[7]_{mod\ 20} = [-13]_{mod\ 20} = ([3]_{mod\ 20})^{-1}.$$

□

Proposition 12.23. (Fermat's little theorem) *Let $p \in \mathbb{Z}^+$ be a* prime *number. For every $a \in \mathbb{Z}$, if $m \equiv n(mod\ (p-1))$, then*

$$a^m \equiv a^n(mod\ p).$$

Example 12.24.

$$2^2 \equiv 1(mod\ 3) \equiv 2^4(mod\ 3) \equiv 2^6(mod\ 3),$$

$$5^2 \equiv 1(mod\ 3) \equiv 5^4(mod\ 3) \equiv 5^6(mod\ 3),$$

$$5^1 = 5 \equiv 2(mod\ 3) \equiv 5^3(mod\ 3) \equiv 5^5(mod\ 3),$$

$$3^4 = 81 \equiv 1(mod\ 5) \equiv 3^8(mod\ 5),$$

$$2^6 = 64 \equiv 1(mod\ 7) \equiv 2^{12}(mod\ 7)$$

$$5^6 = 15.625 \equiv 1(mod\ 7) \equiv 5^{12}(mod\ 7),$$

$$5^2 = 25 \equiv 4(mod\ 7) \equiv 5^8(mod\ 7),$$

$$12^3(mod\ 17) \equiv 12^{19}(mod\ 17).$$

In the notation of Remark 12.22, we have:

$$[3]_{mod\ 5} = [2]^{-1}_{mod\ 5}.$$

Then,

$$[3]^7_{mod\ 5} = [3]^2_{mod\ 5} = [4]_{mod\ 5} = ([2]^{-1}_{mod\ 5})^2 = [2]^{-2}_{mod\ 5} \overset{def}{=} [3]^2_{mod\ 5}.$$

□

12.7.2 The method

RSA (Rivest-Shamir-Adleman) is a public-key cryptosystem that is widely used for secure data transmission. A typical application of public-key cryptography is the *digital signature*.

In a public-key cryptosystem, the *encryption key* is *public* and distinct from the *decryption key*, which is kept *secret (private)*.

An *RSA* user creates and publishes a public key $n = pq$ based on two large prime numbers p, q, along with an auxiliary value. The prime numbers are kept secret. Messages can be encrypted by anyone, via the public key, but can only be decrypted by someone who knows the prime numbers. The security of *RSA* relies on the practical difficulty of factoring the product of two large prime numbers, the *factoring problem*. Breaking *RSA* encryption is known as the *RSA* problem. Whether it is as difficult as the factoring problem is an open question. There are no published methods to defeat the system if a large enough key is used.

Suppose that you have a classified *clear* M message that you need to convey to a friend of yours. How can you do that in a safe way? First, you should build a new message (*encryption*). This process will lead your message to be the encrypted *dark* message $\mathbf{M}$, that you really convey. The next step (*decryption*) is for your friend to rebuild the original clear message M from the dark message $\mathbf{M}$:

$$M \overset{encryption}{\longrightarrow} \mathbf{M}, \quad \mathbf{M} \overset{decryption}{\longrightarrow} M.$$

For the encryption process, we use a known $n \in \mathbb{Z}^+$ key - n being a product $n = pq$ of *two prime numbers* p, q - while for the decryption process we use the evaluation $\Phi(n)$ of the Euler function. In order to calculate it one needs to know the prime numbers p, q. The security of *RSA* relies on the practical difficulty of factoring the product of two large prime numbers, the "factoring problem". In current technology, the numbers n, p, q are of the order of 10^{300}.

We fix the number n and a positive integer e such that

$$1 < e < \Phi(n), \qquad GCD(e; \Phi(n)) = 1;$$

this number e is the *public exponent* and the pair (n, e) is the *public key*.

The encryption process consists in splitting the original clear message into submessages, and by transforming each submessage into a positive integer number m, with the constraint that $1 < m < n$.

Then, we proceed to the encryption of each clear submessage m:

$$(clear)\ m \stackrel{encryption}{\longrightarrow} c \equiv m^e (mod\ n)\ (dark),$$

where c is the unique positive integer $c < n$ such that $c \equiv m^e(mod\ n)$ (Proposition 12.20). In plain words, the dark message c is the remainder in the division by n of the power m^e of the *clear massage* m.

The message m to be encripted may be any positive integer such that $1 < m < n$, but the *public exponent* e, $1 < e < \Phi(n)$, must be *coprime* with $\Phi(n)$.

Now, given the public key (n, e), we have

$$\Phi(n) \ = \ pq\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right) \ = \ (p-1)(q-1),$$

and, from Proposition 12.20,

$$\exists!\ d < \Phi(n)\ such\ that\ d \cdot e \ \equiv \ 1(mod\ \Phi(n)). \tag{68}$$

The pair (n, d) is the *private key* and d is the *private exponent*.

The decryption process is:

$$(dark)\ c \stackrel{decryption}{\longrightarrow} c^d(mod\ n) \ \equiv \ m\ (clear).$$

In plain words, the original message m is the remainder in the division by n of the power c^d of the *dark message* c.

12.7.3 Proof

The *exponents* $e, d < \Phi(n)$ are such that

$$ed \equiv \ 1(mod\ (p-1)(q-1)).$$

Hence,

$$ed \ = \ h(p-1)(q-1) + 1.$$

Thus,

$$ed \equiv \ 1(mod\ (p-1))$$

and

$$ed \equiv \ 1(mod\ (q-1)).$$

From Proposition 12.23, we infer:

$$(m^e)^d = m^{ed} \equiv m(mod\ p) \Longrightarrow m^{ed} - m \equiv 0(mod\ p)$$

and

$$(m^e)^d = m^{ed} \equiv m(mod\ q) \Longrightarrow m^{ed} - m \equiv 0(mod\ q).$$

Then, p *divides* $m^{ed} - m$ and q *divides* $m^{ed} - m$.

Since p, q are *distinct primes*, then

$$n = pq \ \ \textbf{divides} \ \ m^{ed} - m.$$

Therefore,

$$m^{ed} = (m^e)^d = c^d(mod\ n) \stackrel{!}{\equiv} m,$$

is exactly the original *clear* message. □

Example 12.25. Let $n = 25$, then, $\Phi(25) = 20$.

Let $e = 3$ be the *public exponent*; then, the *private exponent* is $d = 7$ (indeed $7 \cdot 3 = 21 \equiv 1(mod\ 20)$).

Let $m = 14$ be the *clear message*. Then, $c = 19 \equiv 14^3(mod\ 25) = m^e(mod\ 25)$ is the *dark message*.

Hence,

$$c^d(mod 25) = 19^7(mod\ 25) = 14 = m,$$

as desired.

□

References

[1] J.P.S. Kung, G.-C. Rota and C.H. Yan, *COMBINATORICS. The Rota Way*, Cambridge Univ. Press, 2009

[2] R. Graham, D. Knuth, O. Patashnik, *CONCRETE MATHEMATICS. A Foundation for Computer Science*, Addison-Wesley, 1994

[3] R.P. Stanley, *Enumerative Combinatorics. Volume* 1, Cambridge Univ. Press, 2011

[4] G.-C. Rota, On the foundations of combinatorial theory I: Theory of Moebius functions, *Zeitschrift fur Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 1964, Vol. 2, pp. 340–368

[5] M. Barnabei, A. Brini, G.-C. Rota, La Teoria delle Funzioni di Moebius (in *russian*), *Uspekhi Matematicheskikh Nauk*, Russian Academy of Sciences and Moscow Mathematical Society, 1986, Vol. 41, pp. 113–157. English version: The Theory of Moebius Functions, *Russian Mathematical Surveys*, 1986, London Mathematical Society and the British Library, Vol. 41, pp. 135–188

Bologna, december 2020