Wednesday, February 14, 2024 11:30 AM

BEGIN AT 17.18

MAIN THEOREM (THM 16, page 24 ff)

$f : E \subseteq \mathbb{R}^n \to \mathbb{R}$, E MEAS

i) f LIMITED + ii) $\mu(E) < +\infty$.

f IS $\mathcal{L}$-INTEGRABLE $\overset{\text{THM}}{\iff}$ f MEASURABLE FUNCTION.

$\Downarrow$

$$\underset{-E}{\int} f = \overset{-}{\underset{E}{\int}} f$$

NOTICE THAT

I RECALL $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$

ADMITS RIEMANN INTEGRAL UNDER WHICH CONDS ???

THM (LEBESGUE / VITALI THM ~ 1910).

$f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ IS RIEMANN INTEGRABLE ($\mathcal{R}$-INTEGRABLE)

IF AND ONLY IF

f LIMITED + CONTINUOUS A.E.

NOTICE THAT

f CONTINUOUS A.E. $\Longrightarrow$ f MEASURABLE

f $\mathcal{R}$-INTEGRABLE $\Longrightarrow$ f $\mathcal{L}$-INTEGRABLE

$\not\Longleftarrow$

EX $f : [0,1] \to \mathbb{R}$ S.T.

$$f(x) = \begin{cases} 1 & x \in \mathbb{R} - \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$$

IS DISCONTINUOUS EVERYWHERE

f IS NOT $\mathcal{R}$-INTEGRABLE, BUT

$f \equiv \chi_{[0,1] \setminus \mathbb{Q}}$ SIMPLE FUNCT (IN THE SENSE

MEAS COUNT $\mu(\emptyset)=0$ OF LEBESGUE)

$\Downarrow$ MEAS $\Downarrow$ STEP 1

$$\int f \overset{DEF}{=} \int \chi_{[0,1]\setminus \mathbb{Q}} = \mu([0,1]\setminus\mathbb{Q}) = 1 \quad !!!$$

PROPERTIES OF THE LEBESGUE INTEGRAL OF STEP 2.

PROP. $f, g : E \subseteq \mathbb{R}^n \to \mathbb{R}$, E MEAS

f, g LIMITED + $\mu(E) < +\infty$ (STEP 2)

LET

HP) f, g MEAS $\overset{\text{THM}}{\Longleftrightarrow}$ f, g L-INTEGRABLE..

THEN

i) (LINEARITY) LET $\alpha, \beta \in \mathbb{R}$

$$\int_E (\alpha \cdot f + \beta \cdot g) = \alpha \cdot \int_E f + \beta \int_E g.$$

ii) (MONOTONICITY)

$$f \leq g \text{ A.E.} \Longrightarrow \int_E f \leq \int_E g$$

iii) (ADDITIVITY) LET $E_1, E_2 \subseteq E$, $E_1 \cap E_2 = \emptyset$, $E = E_1 \dot\cup E_2$

$$\int_E f = \int_{E_1} f + \int_{E_2} f.$$

FURTHER RELATION WITH RIEMANN INTEGRABILITY

RECALL SOMETHING ABOUT R-INTEGRABILITY

FOR A FUNCT $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$, f LIMITED

RECALL

$x_0 = a < x_1 < x_2 < \dots < x_{n-1} < b = x_n$

$(x_0, x_1, \dots, x_{n-1}, x_n)$ IS SAID TO BE

A SUBDIVISION OF THE INTERVAL $[a,b]$.

A STEP FUNCT $\Phi : [a,b] \to \mathbb{R}$ RELATIVE

TO THE SUBDIVISION $(x_0 = a < x_1 < \dots < x_{n-1} < x_n = b)$

IS A FUNCT

$$\Phi = a_0 \chi_{[x_0, x_1]} + \sum_{i=1}^{n-1} a_i \chi_{]x_i, x_{i+1}]} \Leftarrow$$

THE LOWER R-INTEGRAL OF $f : [a,b] \to \mathbb{R}$

$$R \underline{\int_a^b} f(x)\,dx \overset{DEF}{=} \sup_{\substack{\Phi \le f \\ \Phi \text{ STEP}}} \int_{[a,b]} \Phi$$

AND DEF $\displaystyle\int \Phi = a_0(x_1 - x_0) + \sum_{i=1}^{n-1} a_i (x_{i+1} - x_i)$

UPPER RIEMANN INTEGRAL

$$R \overline{\int_a^b} f(x)\,dx \overset{DEF}{=} \inf_{\substack{\Psi \ge f \\ \Psi \text{ STEP}}} \int_{[a,b]} \Psi$$

DEF

f IS R-INTEGRABLE $\overset{DEF}{\iff} R \underline{\int_a^b} f(x)\,dx \overset{!}{=} R \overline{\int_a^b} f(x)\,dx$.

NOTICE THAT

Φ STEP FUNCT $\implies \Phi$ SIMPLE

$f : [a,b] \to \mathbb{R}$ LIMITED , RECALL

$$R \underline{\int_a^b} f(x)\,dx = \sup_{\substack{\Phi \le f \\ \underline{\Phi \text{ STEP}}}} \int \Phi$$

$$\ge \qquad \le$$

$$\underline{\int_{[a,b]}} f = \sup_{\substack{\varphi \le f \\ \underline{\varphi \text{ SIMPLE}}}} \int \varphi \le$$

$$\le \overline{\int_{[a,b]}} f \overset{DEF}{=} \inf_{\substack{\psi \ge f \\ \underline{\psi \text{ SIMPLE}}}} \int \psi$$

$$\le$$

$$R \overline{\int_a^b} f(x)\,dx \overset{DEF}{=} \inf_{\substack{\Psi \ge f \\ \underline{\Psi \text{ STEP}}}} \int \Psi$$

THEN

$$R \underline{\int_a^b} f(x)\,dx \underset{=}{\le} \underline{\int_{[a,b]}} f \underset{=}{\le} \overline{\int_{[a,b]}} f \underset{=}{\le} R \overline{\int_a^b} f(x)\,dx$$

EQUAL!!

f R-INTEGRABLE $\overset{DEF}{\iff} R \underline{\int_a^b} f(x)\,dx = R \overline{\int_a^b} f(x)\,dx$

THEN

f R-INTEGRABLE $\Longrightarrow$ f L-INTEGRABLE

+

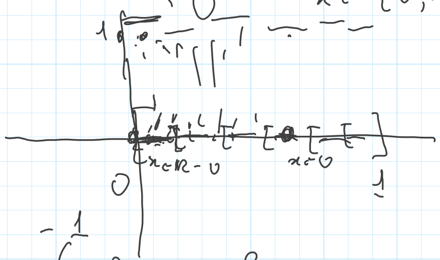
$$R\int_a^b f(x)\,dx = \int_{[a,b]} f$$

<u>BREAK</u>   <u>QUESTIONS ???</u>

<u>RMK</u>

$f:[0,1]\to\mathbb{R}$

$$f(x) = \begin{cases} 1 & x\in[0,1]\setminus\mathbb{Q} \\ 0 & x\in[0,1]\cap\mathbb{Q} \end{cases}$$



$$R\int_0^1 f(x)\,dx \overset{?}{=} 1$$

$\neq \Longrightarrow f$ NOT RIEMANN INTEGRABLE !!!

$$R\int_0^1 f(x)\,dx \overset{?}{=} 0$$

$\cancel{R\int_0 f(x)\,dx}$ WHILE $\int_{[0,1)} \; = \underline{1}$ LEBESGUE