Wednesday, February 14, 2024 11:29 AM

STEP 1 LEBESGUE INTEGRALS FOR SIMPLE FUNCTS.

GIVEN $A \subseteq \mathbb{R}^n$, DEFINE THE CHARACTERISTIC FUNCT OF $A \subseteq \mathbb{R}^n$:

$\chi_A : \mathbb{R}^n \longrightarrow \mathbb{R}$ S.T.

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & \text{OTHERWISE.} \end{cases}$$

RECALL:

PROP.

A MEAS (AS SUBSET) $\Longleftrightarrow$ χ_A MEAS (AS FUNCT)

DEF. $\phi : \mathbb{R}^n \to \mathbb{R}$ SIMPLE FUNCT

IF AND (OR) ONLY IF

$$\phi = \sum_{j=1}^{m} \underset{\in \mathbb{R}}{e_i} \cdot \chi_{E_j} , \qquad (*)$$

WHERE

i) E_j MEAS !!!

ii) $\mu(E_j) < +\infty$.

FROM i) $\Rightarrow$ ϕ MEASURABLE FUNCT !!!

E_j MEAS $\forall j = 1, \dots, m$ $\Longleftrightarrow$ χ_{E_j} MEAS

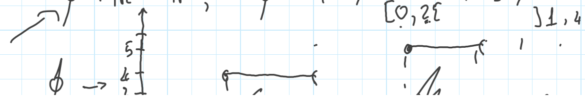
$\Rightarrow \phi = \sum_{j=1}^{m} e_j \cdot \underset{\uparrow \text{MEAS}}{\chi_{E_j}}$

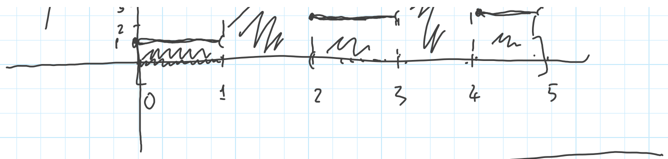
QUESTION: IS PRESENTATION $(*)$ UNIQUE ???

NO !!!

EXAMPLE

$\phi : \mathbb{R} \to \mathbb{R}$, $\phi \overset{?}{=} 1 \cdot \chi_{[0,2[} + 3\chi_{]1,4]} + 2\chi_{]3,5[}$





THEN

$$\phi = 1\cdot\chi_{[0,1[} + 4\,\chi_{[1,2]} + 3\chi_{[3,4[} + 2\,\chi_{[4,5]} \quad (**) \quad \text{CANONICAL !!!}$$

$$= 1\cdot\chi_{[0,2[} + 3\,\chi_{]1,4]} + 2\,\chi_{]3,5]} \quad (*)$$

CANONICAL PRESENTATION

ϕ SIMPLE ,

$$\phi = \sum_{i=1}^{m} e_i \cdot \chi_{E_i} \qquad E_i \text{ MEAS} \quad \mu(E_i) < +\infty$$

IS SAID TO BE CANONICAL

IF AND ONLY IF

1) $e_1, e_2, \dots, e_m \neq 0$ AND

$e_i \neq e_j$ WHENEVER $i \neq j$

2) ALL $E_i \neq \emptyset$ AND

$E_i \cap E_j = \emptyset$ IF $i \neq j$.

CLEARLY , THE CANONICAL PRESENTATION IS UNIQUE !!!

ϕ SIMPLE , AND LET

$$\phi \overset{!}{=} \sum_{i=1}^{m} e_i \,\chi_{E_i} \qquad \text{CANONICAL PRESENTATION}$$

DEFINE THE LEBESGUE INTEGRAL IS (BY DEF.)

$$\int \phi = \sum_{i=1}^{m} e_i \cdot \mu(E_i) \qquad \text{OK}.$$

PROP. (LINEARITY) ϕ, ψ SIMPLE , $\alpha, \beta \in \mathbb{R}$.

THEN $\alpha\cdot\phi + \beta\cdot\psi$ SIMPLE (TRIVIAL)

AND

$$\int (\alpha \cdot \phi + \beta \cdot \psi) = \alpha \int \phi + \beta \int \psi .$$

THEN

$$\phi = \sum_{j=1}^{m} d_j \chi_{A_j} ,$$

$$\int \phi = \sum_{j=1}^{m} d_j \underbrace{\int \chi_{A_j}}_{\mu(A_j)} = \sum_{j=1}^{m} d_j \, \mu(A_j)$$

<u>PROP</u> (<u>MONOTONICITY PROP</u>)

$\phi, \psi : \mathbb{R}^n \to \mathbb{R}$ <u>SIMPLE FUNCTS</u> ,

$\phi \leq \psi$ <u>A.E.</u> $\implies \int \varphi \leq \int \psi$.

<u>STEP 2</u> $f : E \subseteq \mathbb{R}^n \to \mathbb{R}$ <u>E MEAS</u>

S.T. i) f <u>LIMITED</u>

ii) $\mu(E) < +\infty$.

<u>LOWER (LEBESGUE) INTEGRAL</u>:

$$\underline{\int_E} f \overset{\text{DEF}}{=} \sup_{\substack{\varphi \leq f \\ \varphi \text{ SIMPLE}}} \int \varphi$$

<u>DEFINED BY STEP 1</u>

<u>LOWER</u>

<u>NON EMPTY</u>

UPPER (LEBESGUE) INTEGRAL

$$\overline{\int_E} f \overset{DEF}{=} \inf_{\substack{\psi \geq f \\ \psi \text{ SIMPLE}}} \int \psi$$

DEFINED BY STEP 1

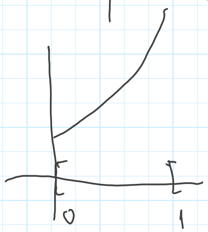
← UPPER

NONEMPTY

WHY: f LIMITED ???

IF, FOR INSTANCE, f IS NOT UPPER LIMITED

$\exists\ \psi$ SIMPLE S.T. $\psi \geq f$???



$\nexists\ \psi$ SIMPLE S.T. $\psi \geq f$

FURTHERMORE

$$\underset{\text{SIMPLE}}{\varphi} \leq f \leq \underset{\text{SIMPLE}}{\psi} \Rightarrow$$

$$\Rightarrow \varphi \leq \psi \overset{\text{STEP1}}{\Longrightarrow} \int \varphi \leq \int \psi$$

$$\underline{\int_E} f = \sup_{\substack{\varphi \leq f \\ \varphi \text{ SIMPLE}}} \int \varphi \leq \inf_{\substack{\psi \geq f \\ \psi \text{ SIMPLE}}} \int \psi = \overline{\int_E} f$$

GENERAL FACT

INTEGRABILITY (LEBESGUE) OF

$f : E \subseteq \mathbb{R}^n \to \mathbb{R}$ E MEAS

i) f LIMITED

ii) $\mu(E) < +\infty$.

DEF f IS LEBESGUE INTEGRABLE

IF AND ONLY IF (DEF)

$$\underline{\int_E} f \stackrel{!}{=} \overline{\int_E} f$$

BREAK QUESTIONS?

BEGIN AGAIN AT 17.18 !!