Wednesday, February 14, 2024 11:31 AM

STEP 2 $f: E \subseteq \mathbb{R}^n \to \mathbb{R}$ E MEAS

i) f LIMITED , ii) $\mu(E) < +\infty$.

RECALL THM (MAIN THM)

f L-INTEGRABLE $\iff$ f MEASURABLE.

RECALL $(f_n)_{n \in \mathbb{N}}$, $f_n : E \subseteq \mathbb{R}^2 \to \mathbb{R}$

WE HAVE:

1) $f_n \xrightarrow[n \to \infty]{P.W.} f : E \to \mathbb{R} \overset{DEF}{\iff}$

$\iff \forall x \in E \ (\forall \varepsilon \in \mathbb{R}^+ \ \exists \nu_{x,\varepsilon} \in \mathbb{N}$ t.c.

$|f(x) - f_n(x)| < \varepsilon \ \ \forall n > \nu_{x,\varepsilon})$

$\iff \forall x \in E \ (f_n(x) \xrightarrow[n \to \infty]{} f(x))$

2) $f_n \xrightarrow[n \to \infty]{UNIF.} f \overset{DEF}{\iff}$

$\forall \varepsilon \in \mathbb{R}^+ \ \exists \nu_\varepsilon \in \mathbb{N}$ s.t.

$|f(x) - f_n(x)| < \varepsilon \ \forall n > \nu_\varepsilon, \ \forall x \in E$.

WHAT ABOUT $(f_n)_{n \in \mathbb{N}}$, $f_n : E \subseteq \mathbb{R}^n \to \mathbb{R}$,

E MEAS, f_n MEAS $\forall n$, f_n LIMITED + $\mu(E) < +\infty$ } (*)

(*) MEANS THAT ARE WITHIN STEP 2 !!!

WE CAN SAY: (QUOTATION FROM LITTLEWOOD ~1920)

" EVERY PW CONVERGENT SEQUENCE OF FUNCT (*)

IS ALMOST UNIF. CONVERGENT !!! " (?)

THM (EGOROFF THM) $f_n : E \subseteq \mathbb{R}^n \to \mathbb{R}$,

f_n MEAS + LIMITED AND $\mu(E) < +\infty$.

ASSUME THAT $(f_n)_{n \in \mathbb{N}}$ IS S.T.

$f_n \xrightarrow[n \to \infty]{P.W.} f$.

THEN $\forall \eta \in \mathbb{R}^+$ (ARBITRARILY "SMALL")

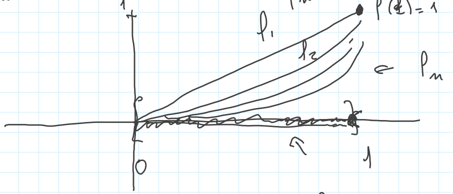
$\exists B \subseteq E$, B MEAS WITH $\mu(B) < \eta$.

SUCH THAT

$$f_n \xrightarrow[n\to\infty]{\text{UNIF.}} f \qquad \text{ON} \quad \underline{E \setminus B} \quad !!!$$

<u>EX/COUNTEREXAMPLE</u> (ABOUT THE EGOROFF THM)

$f_n : [0,1] \to \mathbb{R}$, $\quad f_n(x) = x^n$ $\quad , x \in [0,1]$



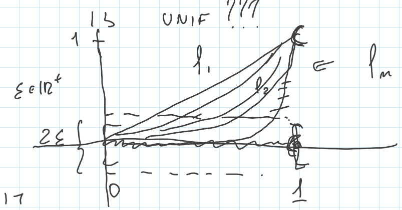
$$f_n \xrightarrow[n\to\infty]{\text{P.W.}} f \quad , \text{WHERE} \quad f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$$

BUT $f_n \cancel{\xrightarrow[n\to\infty]{\text{UNIF}}} f$.

WE CAN WONDER: IF WE REMOVE THE POINT $\underline{x=1}$

THE CONVERGENCE ON $[0,1] \setminus \{1\} = [0,1[$

IS UNIF ???



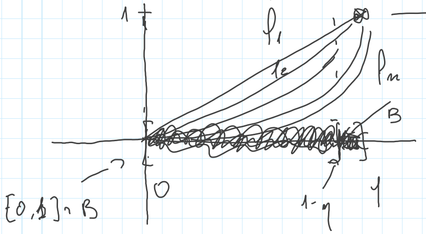
$B = \{1\}$

$E \setminus B = [0,1[$

$f \equiv 0$ ON $[0,1[$.

IS IT TRUE THAT $f_n \xrightarrow[n\to\infty]{\text{UNIF}} f \equiv 0$ ON $[0,1[$??? NO

TRUE MEANING OF <u>EGOROFF THM</u> !!!



$$f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$$

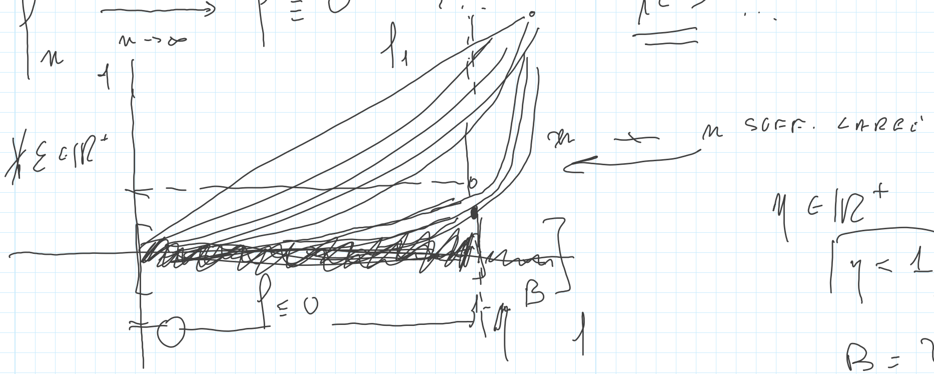
FIX ANY $\eta \in \mathbb{R}^+$ AND CONSIDER

$$B = \,]1-\eta, 1] \implies \mu(B) = \eta.$$

CONSIDER $\underline{f_n \xrightarrow[n\to\infty]{\text{P.W.}} f \equiv 0}$ ON $\underline{[0,1] \setminus B}$

IS IT TRUE THAT

$f_n \xrightarrow[n\to\infty]{\text{UNIF}} f \equiv 0$??? **YES !!!**



$\eta \in \mathbb{R}^+$, $\eta < 1$

$B =]1-\eta, 1]$

$f_n \xrightarrow[n\to\infty]{\text{UNIF}} f \equiv 0$ ON $[0,1] \setminus B$

IN A FORMAL WAY:

$f_n(x) = x^n$ IS MONOTONE INCREASING FUNCT ON $[0,1]$

$$\sup_{x \in [0,1]\setminus B} f_n(x) = f_n(1-\eta) = (1-\eta)^n$$

SINCE $1-\eta < 1 \implies (1-\eta)^n \xrightarrow[n\to\infty]{} 0$

$\implies$

$\forall \varepsilon \in \mathbb{R}^+ \quad \exists \nu_\eta \in \mathbb{R}^+$ S.T.

$$(1-\eta)^n < \varepsilon \quad \forall n > \nu_\eta$$

$$\| \quad f_n(1-\eta) \quad \|$$

$$\sup_{x \in [0,1]\setminus B} f_n(x) = \sup_{x \in [0,1]\setminus B} |f(x) - f_n(x)| < \varepsilon$$

$\quad\quad \| \; 0 \qquad \forall n > \nu$

$\forall\, n > \nu_\eta$

THAT MEANS THAT

$$f_n \xrightarrow[n\to\infty]{\text{UNIF}} f \equiv 0 \quad \text{ON } [0,1] \setminus B \qquad B =]1-\eta, 1[$$

WHERE $\mu(B) = \eta$

IS IT CLEAR ? QUESTIONS?

THM (LITTLEWOOD LEMMA) $f_n : E \subseteq \mathbb{R}^n \to \mathbb{R}$ E MEAS

$\mu(E) < +\infty$, f_n LIMITED + MEAS.

ASSUME THAT $f_n \xrightarrow[n\to]{P.W.} f$ ON $E \subseteq \mathbb{R}^n$ HP

THEN $\forall\, \varepsilon \in \mathbb{R}^+$, $\forall\, \delta \in \mathbb{R}^+$ $\exists\, N \in \mathbb{N}$

$\exists\, B \subseteq E$, B MEAS, $\mu(B) \leq \delta$

S.T.

$$|f(x) - f_n(x)| < \varepsilon \quad \forall\, x \notin B \quad \forall\, n > N.$$

BREAK QUESTIONS???

BEGIN AGAIN AT ~17.10