Wednesday, February 14, 2024 11:31 AM

BEGIN AGAIN AT ~17.10

DOMINATED P.W LIMIT THM FOR LEBESGUE INTEGRALS OF STEP 2. (f_n LIMITED, $\mu(E) < +\infty$, f_n MEAS).

THM $(f_n)_{n \in \mathbb{N}}$, $f_n : E \subseteq \mathbb{R}^n \to \mathbb{R}$ f_n LIMITED, f_n MEAS

AND $\mu(E) < +\infty$.

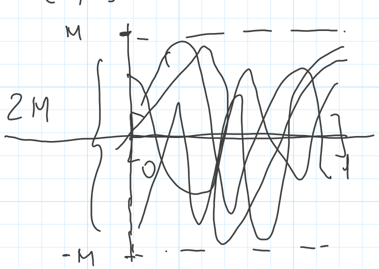
IF HP 1) $f_n \xrightarrow[n\to\infty]{P.W.} f$ ($\Rightarrow f$ LIMITED AND MEAS)

HP 2) (DOMINANCE COND)

$\exists\ M \in \mathbb{R}^n$ S.T.

$|f_n(x)| \leq M \quad \forall x \in E \ \forall n \in \mathbb{N}$.

(EX. $E = [0,1] \subseteq \mathbb{R}$



$f_n \quad \forall n \in \mathbb{N}$)

THEN

(*) $\int_E f_n \xrightarrow[n\to\infty]{} \int_E f$??? ←

THIS IS A CONV. SEQUENCE

EXISTS! WHY

PROOF BY LITTLEWOOD LEMMA.

$\forall\ \varepsilon \in \mathbb{R}^+ \quad \exists\ N \in \mathbb{N} \quad \exists\ B \subseteq E$, B MEAS

S.T. $\mu(B) < \frac{\varepsilon}{4M} \quad \left(\eta = \varepsilon/4M \right)$

S.T.

(**) $\left|f_n(x) - f(x)\right| < \frac{\varepsilon}{2\cdot\mu(E)}$ $\quad \left(\varepsilon = \frac{\varepsilon}{2\mu(E)}\right)$

$\forall\, x \in E - B \quad \forall\, n > N$

THEN

$$\left|\int_E f_n - \int_E f\right| = \left|\int_E (f_n - f)\right| \overset{\text{MONOTONICITY}}{\leq}$$

$$\leq \int_E |f_n - f| \overset{\text{ADD.}}{=} \underbrace{\int_{E-B} |f_n - f|}_{(\dagger)} + \underbrace{\int_B |f_n - f|}_{(\dagger\dagger)}$$

BUT NOW

(†) $\int_{E\setminus B} |f_n - f| \leq \frac{\varepsilon}{2\cdot\mu(E)}\cdot\mu(E\setminus B) \leq \frac{\varepsilon}{2\cdot\mu(E)}$

$\mu(E\setminus B) \leq \mu(E)$ SINCE $E\setminus B \subseteq E$

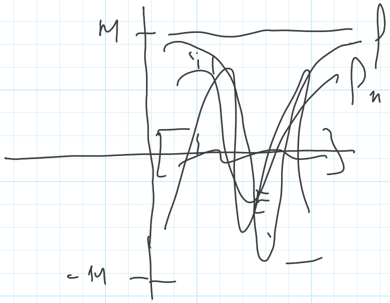
(††) $\int_B |f_n - f| \leq 2M\cdot\mu(B) =$

$\leq 2M \quad \leq 2M\cdot\frac{\varepsilon}{2M}$

NOW SINCE $= |f_n(x)| < M \quad \forall x \in B \subseteq E \quad \forall n \in \mathbb{N}$

$\leq \varepsilon/2$

(***) $|f_n - f| < 2M$ ON B $\Longleftarrow$



WE PROVED THAT:

$\underline{\forall\ \varepsilon \in \mathbb{R}^+}$ $\exists\ N \in \mathbb{N}$ $\exists\ B \in \mathcal{E}$ $\mu(B) = \varepsilon / 4M$

$$\underline{\left| \int_E f_n - \int_E f \right|} \leq$$

$$< \underbrace{\int_{E \setminus B} |f_n - f|}_{(+)} + \underbrace{\int_B |f_n - f|}_{(++)} \leq$$

$$< \varepsilon/2 + \varepsilon/2 < \varepsilon \qquad \forall\ \underline{\underline{n > N}}.$$

THAT IS

$$\int_E f_n \xrightarrow[n \to \infty]{} \int_E f$$

$\underline{\underline{\text{QED}}}$

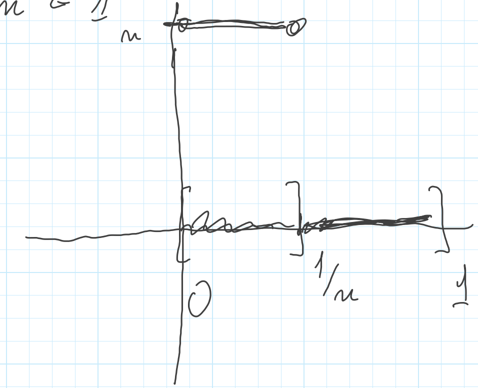
<u>COUNTER EXAMPLE</u> (ABOUT DELETING THE "DOMINANCE CONDITION").

CONSIDER

$f_n : [0,1] \to \mathbb{R}$, $\forall n \in \mathbb{Z}^+$

IS

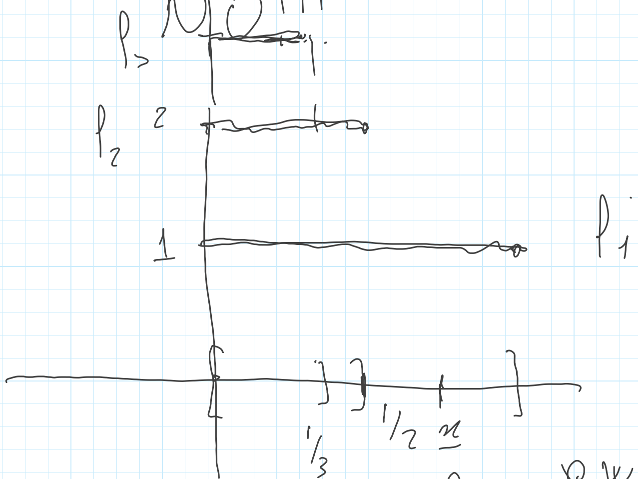
$f_n = n \cdot \chi_{[0,\frac{1}{n}]}$



CLEARLY, f_n LIMITED $\forall n \in \mathbb{Z}^+$

IS THE SEQUENCE $(f_n)_{n \in \mathbb{Z}^+}$ DOMINATED?

NO!!!



CLEARLY $f_n \xrightarrow[n\to\infty]{p.w.} f \equiv 0$!!!

THEN $\int_{[0,1]} f = 0$!!!

BUT

$$n \in \mathbb{Z}^+ \quad \int_{[0,1]} f_n \overset{\text{DEF}}{=} \int_{[0,1]} n \cdot \chi_{[0,\frac{1}{n}]} = n \cdot \mu\left[0,\tfrac{1}{n}\right] = n \cdot \tfrac{1}{n} = \underline{1}$$

$$\forall n \in \mathbb{Z}^+ \quad \int_{[0,1]} f_n = 1 \xrightarrow[n \to \infty]{} \underline{1} \neq 0 = \int_E f$$

SO, THE ASSERTION IS FALSE.

STOP QUESTIONS ???

BYEBYE, SEE YOU TOMORROW !!