Wednesday, February 14, 2024 11:32 AM

STEP 2 $f : E \subseteq \mathbb{R}^n \to \mathbb{R}$, E MEAS

f ~~LIMITED~~ + $\mu(E)$ ~~$< +\infty$~~ .

MAIN THM f L-INTEGRABLE $\iff$ f MEASURABLE.

STEP 3 $f : E \subseteq \mathbb{R}^n \to \mathbb{R}$ E MEAS

SUPPOSE f IS NON NEGATIVE $\iff$ $f(x) \geq 0 \quad \forall x \in E$.

LET $f : E \subseteq \mathbb{R}^n \to \mathbb{R}$,

f MEASURABLE + f N.N.

$$\int_E f \overset{\text{DEF}}{=} \sup_{h \leq f} \int_E h \qquad (*)$$

WHERE h S.T. :

i) h MEAS (+ h N.N.)

ii) h LIMITED

iii) $\text{supp}(h) = \{x \in E \ ; \ h(x) \neq 0\}$

IS S.T. $\mu(\text{supp}(h)) < +\infty$.

$$\int_E h = \int_{\text{supp}(h)} h \qquad (\dagger)$$

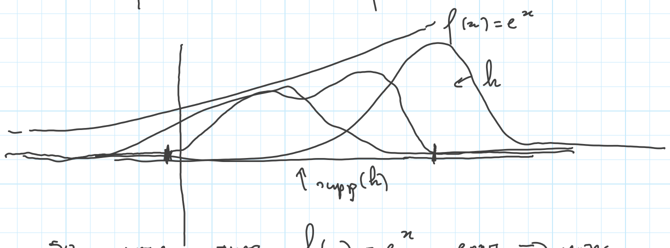
THESE ARE ALL DEFINED BY STEP 2

$$(\dagger) \quad \int_E f = \sup_{h \leq f} \int_E h = \sup_{h \leq f} \int_{\text{supp}(h)} h$$

CONSISTENT

EX $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$

$(\dagger)$

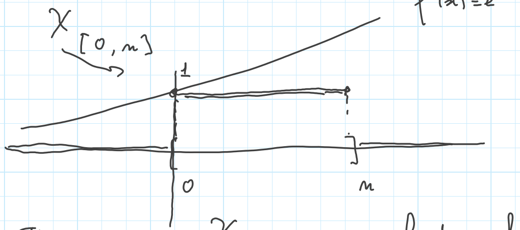


SO, NOTICE THAT $f(x) = e^x$ CONT. $\implies$ MEAS

AND f N.N.

NOTICE THAT $\int_{\mathbb{R}} f = \int_{\mathbb{R}} e^x = +\infty$.

WHY? CONSIDER, $\forall n \in \mathbb{Z}^+$



THAT IS $\chi_{[0,n]}$ are functions of type $\to h$

INVOLVED IN THE DEFINITION

$$\overset{+\infty}{\overset{\parallel}{\sup_{n\in\mathbb{Z}^+} n}} = \sup_{n\in\mathbb{Z}^+} \int_{\mathbb{R}} \chi_{[0,n]} \le \qquad \int_{\mathbb{R}} \chi_{[0,n]} = n$$

$$\le \underline{\sup_{h\le f} \int_{supp(h)} h} \overset{DEF}{=} \int_{\mathbb{R}} f$$

THEN $\int_{\mathbb{R}} f = \underline{\underline{+\infty}}$!!! ...

FATOU LEMMA (THM)

$(f_n)_{n\in\mathbb{N}}$, $f_n : E \subseteq \mathbb{R}^d \to \mathbb{R}$ E MEAS

f_n MEAS + N.N.

HP $f_n \xrightarrow[n\to\infty]{P.W.} f$ CLEARLY f MEAS + N.N.

THEN

$$\int_E f \le \underset{n}{\text{min lim}} \int_E f_n \quad !!! \ldots$$

CANNOT BE IMPROVED

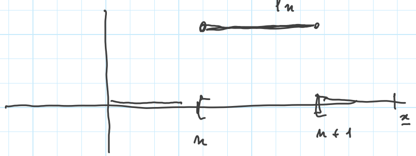
RMK THIS STATEMENT IS THE "BEST POSSIBLE"

COUNTEREXAMPLES $(f_n)_{n\in\mathbb{Z}^+}$.

i) $f_n = \chi_{[n,n+1]}$ $\forall n \in \mathbb{Z}^+$

$$f_n \overset{\text{DEF}}{=} \chi_{[n,n+1[} \xrightarrow[n\to\infty]{\text{P.W.}} f \quad ???$$

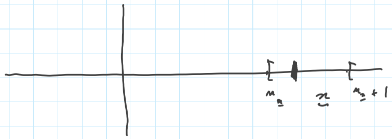
FIXED $n \in \mathbb{Z}^+$



FIX $x \in \mathbb{R}$ STUDY

$$(f_n(x))_{n\in\mathbb{Z}^+} = \left(\chi_{[n,n+1[}(x)\right)_{n\in\mathbb{Z}^+}$$

FIXED $x \in \mathbb{R}^+$



$\exists!\ n_x \in \mathbb{Z}^+$ s.t. $x \in [n_x, n_x+1[$

$$(0,0,\dots,0,\underset{n_x}{1},0\dots\ 0) \xrightarrow[n\to\infty]{} 0\ (x)$$

TRUE FOR ANY $x \in \mathbb{R} \implies$

$$f_n = \chi_{[n,n+1[} \xrightarrow[n\to\infty]{\text{P.W.}} \underline{\underline{f \equiv 0}} \quad \text{THAT IS } f(x) = 0\ \forall x \in \mathbb{R}$$

THEN $\displaystyle\int_{\mathbb{R}} f = 0$.

BUT $\displaystyle\int_{\mathbb{R}} f_n = \int_{\mathbb{R}} \chi_{[n,n+1[} \overset{\text{DEF}}{=} 1$

$$\int_{\mathbb{R}} f_n = (1, 1 \dots, 1 \dots)$$

$$\lim_{n\to\infty} \int f_n = 1 = \min_n \lim \int_{\mathbb{R}} f$$

$$\int_{\mathbb{R}} f = 0 \ \underset{\leq}{<}\ 1 = \min_n \lim \int_{\mathbb{R}} f$$

2) $f_n : [0, +\infty[\longrightarrow \mathbb{R}$

$$f = \nearrow \chi_{[n,n+1[} \quad n \ \underline{\text{even}}$$

$f_n \equiv 0$ n ODD.

$$f_n \xrightarrow[n\to\infty]{P.W} f \equiv 0 .$$

BUT, WHAT IS

$$\left(\int_{[0,+\infty[} f_n \right)_{n \in \mathbb{Z}^+} = (0,1,0,1,0,\dots,1,0\dots)$$

$$\min\lim_n (0,1,0,1\dots) =$$

$$\sup_n \Big(\inf_{k\geq n} \underbrace{(\overset{\text{STARTING FROM } n}{0,1\dots})}_{=0} \Big) = \sup_{n\in\mathbb{N}} (0,0,\dots 0\dots) = 0 = \dots$$

$$\max\lim_n (0,1,0,1\dots) \overset{DEF}{=}$$

$$\inf_n \Big(\sup_{k\geq n} \underbrace{\Big(\int_{\mathbb{R}} f_k \Big)}_{=1} \Big) = \inf_n (1,\dots,1\dots,1) = \underline{1}$$

THEN $\min\lim_n \left(\int_{\mathbb{R}} f_n \right) = 0 \not\leq$

$$\not\leq 1 = \max\lim_n \left(\int_{\mathbb{R}} f_n \right) \implies$$

$\left(\int_{\mathbb{R}} f_n \right)$ IS <u>NOT</u> A <u>CONVERGENT SEQUENCE</u>

$$\int_E f = 0 \neq 0 = \min\lim_n \int_E f_n$$

3) IF $\cap$ REPLACE

$\to f_n$ MEAS $+$NN (TRUE)

BY

$f_n \xrightarrow[n\to\infty]{P.W} f$ WHERE f <u>MEAS $+$NN</u>!

<u>IS IT STILL TRUE THE STATEMENT</u>? <u>NO</u>

<u>COUNTEREXAMPLE</u>

$$f_n = (-1)^n \chi_{[n,n+1[}$$

f_n ARE NOT <u>NN</u> FOR n ODD.

BUT $f \xrightarrow{P.W} f \equiv 0$

$$| _n \quad n\to\infty \quad |$$

$$\int_{\mathbb{R}} f = 0 \quad .$$

BUT $$\int_{\mathbb{R}} f_n = \begin{cases} 1 & \text{EVEN} \\ -1 & \text{ODD} \end{cases}$$

THEREFORE

$$\left(\int_{\mathbb{R}} f_n \right)_{n \in \mathbb{N}} = \left(-1, 1, -1, \dots \dots \right)$$

$$\Downarrow$$

$$\min\lim_{n} \int_{\mathbb{R}} f_n = -1 \qquad \text{THEN}$$

$$\int_{\mathbb{R}} f = 0 \not\leq -1 = \min\lim_{n} \int_{\mathbb{R}} f_n$$

<u>BREAK</u> QUESTIONS?