Wednesday, February 14, 2024 11:33 AM

BEGIN AGAIN AT ~10.15

PROOF OF THE FATOU LEMMA

HP $f_n \xrightarrow[n\to\infty]{\text{PW}} f$ $\qquad f_n$ MEAS + NN

$\Downarrow$

f MEAS + NN

WE RECALL

$$\int_E f = \sup_{h \le f} \int_E h$$

WHERE
i) h MEAS (NN)
ii) h LIMITED
iii) $\mu(\operatorname{supp}(h)) < +\infty$ $\Big\}$ $(*)$

FIX A FUNCT h OF TYPE $(*)$.

SET $\qquad \forall n \in \mathbb{Z}^+$

$$h_n = \underline{\min(h, f_n)} \Rightarrow (\le h)$$

CLEARLY SINCE h LIMITED $\Rightarrow h_n$ LIMITED

FURTHERMORE SINCE

$f_n \xrightarrow[n\to\infty]{\text{P.W.}} f$ AND $h \le f$

$\Downarrow$

$h_n \xrightarrow[n\to\infty]{\text{P.W.}} h$ $\qquad$ (CORRECT THE TYPO AT PAGE 30 f REPLACED BY h)

BUT LET $M = \sup_{x\in E} h(x) \in \mathbb{R}$ SINCE h IS LIMITED

SO $\quad |h_n(x)| \le M \quad \forall n \in \mathbb{Z}^+ \ \forall x \in E$

THEREFORE, THE SEQUENCE $(h_n)_{n\in\mathbb{Z}^+}$

DOMINATED BY M.

HENCE, THE DOMINATED P.W. THM OF STEP 2 HOLDS :

THAT IS

$$(*) \quad \int_E h = \lim_{n\to\infty} \int_E h_n \overset{\text{TRIVIAL}}{=} \min\lim_{n\to\infty} \int_E h_n \le$$

SINCE $h_n \le f_n$ BY DEF $\Rightarrow \int_E h_n \le \int_E f_n$

$$\le \min\lim_{n} \int_E f_n$$

(*) TRUE FOR ANY FUNCT

h MEAS, h LIMITED, $\underline{h \leq f}$

(*) IMPLIES

$$\underline{\int_E f \overset{\text{DEF}}{=} \sup_{\substack{h \leq f \\ h \text{ RIGHT TYPE}}} \int_E h} \leq \min\lim_n \int_E f_n \quad ??? $$

$\underline{QED}$.

$\underline{\text{THM}}$ (BEPPO LEVI "$\underline{\text{MONOTONE CONV THM}}$")

$(f_n)_{n \in \mathbb{N}}$, $f_n : E \subseteq \mathbb{R}^n \to \mathbb{R}$ $\underline{\text{MEAS + NN}}$.

$\underline{\text{MONOTONICITY HP}}$

$f_n \leq f_{n+1}$ $\forall n \in \mathbb{N}$.

$\underline{\text{SET}}$ (?)

$$\underline{f = \lim_{n\to\infty}^{\text{PW}} f_n} .$$

THEN

$$\lim_{n\to\infty} \int_E f_n \overset{!}{=} \int_E f .$$

$\underline{\text{WHY}}$ (?) $\underline{\text{RMK}}$

$$f_n \leq f_{n+1} \implies f_n \xrightarrow[n\to\infty]{\text{PW}} f \ (?)$$

$\underline{\text{HP}}$ $\underline{f_n \leq f_{n+1}}$ $\forall n \in \mathbb{N}$

$$\min\lim_n f_n \overset{\text{DEF}}{=} \sup_n \Big(\underset{\substack{= \\ f_n}}{\inf_{k \geq n} f_k} \Big) = \sup_n f_n \quad !!!$$

$$\max\lim_n f_n \overset{\text{DEF}}{=} \inf_n \Big(\underset{\substack{= \\ \sup_n f_n}}{\sup_{k \geq n} f_k} \Big) = \sup_n f_n \quad !!!$$

$$\min\lim_n f_n = \sup_n f_n = \max\lim_n f_n \implies$$

$$\exists \lim_{n\to\infty} f_n = \sup_n f_n .$$

IN OUR SITUATION (BEPPO LEVI THM)

$f \in f_{n+1} \ \forall n \implies \exists \lim_{n\to\infty} f = f = \sup_n f_n$.

NOTATION

THEN

$f_n \leq f \ (= \sup_n f_n) \ \forall n \xrightarrow{\text{MONOTONICITY}}$

$\int_E f_n \leq \int_E f \quad \forall n \implies$

$$\max\lim_n \int_E f_n \underset{=}{\leq} \int_E f \overset{\text{FATOU LEMMA}}{\underset{=}{\leq}} \min\lim_n \int_E f_n \qquad (*)$$

BUT, IN GENERAL

$$\min\lim_n \int_E f_n \leq \max\lim_n \int_E f_n \qquad (**)$$

SO (*) AND (**) IMPLY

$$\max\lim_n \int_E f_n \overset{!}{=} \int_E f \overset{!}{=} \min\lim_n \int_E f_n$$

$$\Rightarrow \min\lim_n \int_E f_n \overset{!}{=} \max\lim_n \int_E f \overset{\text{DEF}}{=}$$

$$\overset{!}{=} \lim_{n\to\infty} \int_E f_n \overset{!}{=} \int_E f \qquad \underline{\text{QED}}$$

BEPPO LEVI THM (SECOND VERSION)

$(f_n)_{n\in\mathbb{N}}$, f_n MEAS + NN.

ASSUME
- HP 1: $f_n \xrightarrow[n\to\infty]{\text{P.W.}} f$
- HP 2: $f_n \leq f \quad \forall n \in \mathbb{Z}^+$

THEN

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

THE PROOF IS THE SAME

STOP QUESTIONS?