Wednesday, February 14, 2024 11:35 AM

BEPPO LEVI THM (MONOTONE CONV. VERSION)

THM $(f_n)_{n \in \mathbb{N}}$, $f_n : E \subseteq \mathbb{R}^n \to \mathbb{R}$ MEAS + NN.

SUPPOSE $f_n \le f_{n+1}$ $\forall n \in \mathbb{N}$. $\Leftarrow$

LET $f \overset{\text{not}}{=} \overset{\text{p.w.}}{\lim_{n\to\infty}} f_n$ $\left(\overset{\text{mon}}{=} \sup_n f_n \right)$.

THEN

$$\int_E f \overset{!}{=} \lim_{n\to\infty} \int_E f_n \quad !!!$$

A TYPICAL APPLICATION.

LET $f : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ $\quad f$ MEAS + $\boxed{\text{NN}}$.

PROBLEM COMPUTE $\int_E f$???

CONSIDER A NESTED SEQUENCE OF SUBSETS

(o) $(E_n)_{n \in \mathbb{Z}^+}$, $E_n \subseteq E$, E , E_n MEAS $\forall n \in \mathbb{Z}^+$

S.T.

i) $E_n \subseteq E_{n+1}$ $\forall n$

ii) $\bigcup_{n \in \mathbb{Z}^+} E_n = E$.

NOW, SET IS NN !!!

(†) $f_n \overset{\text{DEF}}{=} f \cdot \chi_{E_n}$.

HENCE

$$\boxed{f_n \le f_{n+1} \quad !!!}$$

INDEED:

$$f_n(x) = \begin{cases} f(x) \ge 0 & x \in E_n \\ 0 & x \notin E_n \end{cases}$$

$$f_{n+1}(x) = \begin{cases} f(x) \geq 0 & x \in E_{n+1} \\ 0 & x \in E_n \end{cases}$$

SO, IF $x \in E_n \Rightarrow f_n(x) = f_{n+1}(x) \geq 0$

IF $x \in E_{n+1} \setminus E_n \Rightarrow f_n(x) = 0 \leq f_{n+1}(x) \geq 0$

IF $x \notin E_{n+1} \Rightarrow f_n(x) = 0 = f_{n+1}(x)$.

FURTHERMORE, SINCE

$$\bigcup_n E_n = E \Longrightarrow$$

$$f_n \stackrel{DEF}{=} f \cdot \chi_{E_n} \xrightarrow[n\to\infty]{P.W.} f \quad \geq 0$$

HENCE $f_n \stackrel{DEF}{=} f \cdot \chi_{E_n}$ SATISFIES THE BEPPO LEVI HPS

$$\Downarrow$$

$$\int_E f \stackrel{THM}{=} \lim_{n\to\infty} \int_E f_n \stackrel{DEF}{=} \lim_{n\to\infty} \int_E f \cdot \chi_{E_n} =$$

$$= \lim_{n\to\infty} \int_{E_n} f \quad \leftarrow \text{COMPUTED IN A "SIMPLE WAY"}$$

EX 1 LET $f :]1, +\infty[\to \mathbb{R}$,

$f(x) = 1/x$. (CONT $\Rightarrow$ MEAS) + NN.

$f(x) = 1/x$

COMPUTE $\int f = \int 1/x$???

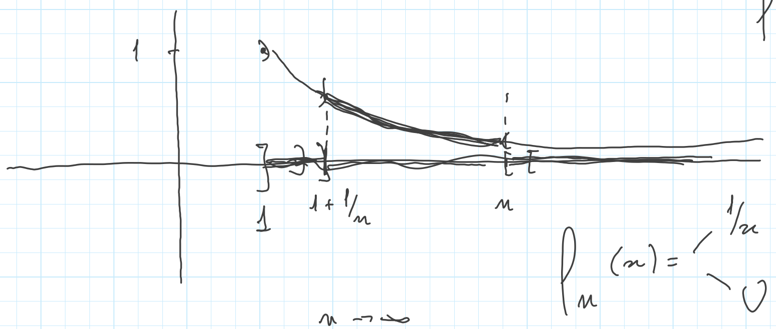
$E =]1,+\infty[\quad]1,+\infty[$

FOR EVERY $n \in \mathbb{Z}^+$, LET

$$E_n =]1+1/n, n[$$ AND SET

$$f_n = f \cdot \chi_{E_n}.$$

$$f(x) = 1/x$$



$$f_n(x) = \begin{cases} 1/x, & x \in]1+1/n, n[\\ 0 & \text{OTHERWISE} \end{cases}$$

NOTICE THAT ; i) $E_n \subseteq E_{n+1} \quad \forall n$

ii) $\bigcup_{n \in \mathbb{Z}^+} E_n =]1,+\infty[= E.$

THEN

$$\int_{]1,+\infty[} f \overset{\text{DEF}}{=} \int_{]1,+\infty[} 1/x \overset{\text{THM}}{=} \lim_{n\to\infty} \int_{]1,+\infty[} f \cdot \chi_{E_n} \overset{!}{=}$$

$$= \lim_{n\to\infty} \underline{\int_{]1+1/n, n[} f}$$

$$\overset{!}{=} \lim R \int^{n} 1/x \cdot dx \qquad \swarrow \text{RIEMANN INTEGRAL!}$$

$n\to\infty$ $\quad 1+1/n$

$$= \lim_{n\to\infty} \Big[\log x \Big]_{x=1+1/n}^{x=n} =$$

$$= \lim_{n\to\infty} \Big[\log n - \log\left(1+1/n\right) \Big]$$

$$= +\infty - 0 = +\infty \quad !!!$$

WE PROVED : $f(x) = 1/x$

$$\int_{]1,+\infty[} f = +\infty$$