Wednesday, February 14, 2024 11:35 AM

BEGIN AGAIN AT 17.10

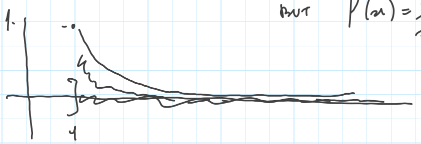
QUESTIONS???

EX 2 LET $f :]1,+\infty[\to \mathbb{R}$,

$$f(x) = \frac{1}{x^m} , \quad m \in \mathbb{Z} , \quad m > 2 \quad !!!$$

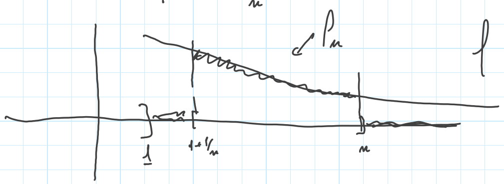
COMPUTE $\displaystyle\int_{]1,+\infty[} f$???

BUT $f(x) = \frac{1}{x^m}$, $m \geq 2$.



LET $E_n = [1 + 1/n , n]$ $\quad n \in \mathbb{Z}^+$

AND $f_n = f \cdot \chi_{E_n}$.



$E_n \subseteq E_{n+1}$ $\forall n$ & $\bigcup_n E_n = E =]1,+\infty[$

THEN

$$\int_{]1,+\infty[} f = \lim_{n\to\infty} \int_{]1,+\infty[} f_n = \lim_{n\to\infty} \int_{E_n = [1+1/n , n]} f =$$

$$= \lim_{n\to\infty} \int_{E_n = [1+1/n , n]} \frac{1}{x^m} = \qquad \boxed{m \geq 2}$$

$$= \lim_{n\to\infty} R \underbrace{\int_{1+1/n}^{n} \frac{1}{x^m}\, dx}_{} =$$

$$= \lim \left[- \frac{1}{\ } \ \frac{1}{\ } \right]^{x=n}$$

$$= \lim_{n\to\infty} \left[\quad \frac{\quad}{m-1} \; \frac{\quad}{x^{m-1}} \right]_{x=1+1/n}$$

$$= \lim_{n\to\infty} \left[-\underbrace{(m-1)^{-1}\cdot n^{-m+1}}_{\to 0} - (m-1)^{-1}\underbrace{\left(1+\tfrac{1}{n}\right)^{m-1}}_{\to 1} \right] =$$

$$= 0 - \left(-(m-1)^{-1}\right) = \underline{\frac{1}{m-1}} < +\infty$$

IN PARTICULAR, $m = 2 \;\Rightarrow\; \frac{1}{m-1} = 1$

$$\int_{]1,+\infty[} \frac{1}{x^2} = 1 < +\infty \qquad !!! \; \ldots$$

<u>IS IT CLEAR?</u> <u>QUESTIONS?</u>

<u>BEPPO LEVI THM FOR SERIES</u>

LET $(f_n)_{n\in\mathbb{N}}$, $f_n : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ <u>MEAS + NN</u>.

CONSIDER, THE SERIES

$$(*) \quad \sum_{n=0}^{\infty} f_n \qquad !!! \; \ldots$$

RECALL THAT

$$(*) \overset{DEF}{=} \underbrace{\left(\sum_{k=0}^{n} f_k\right)_{n\in\mathbb{N}}}_{\text{SERIES}} = \underbrace{\left(f_0,\, f_0+f_1,\, f_0+f_1+f_2,\, \ldots\right)}_{\text{SEQUENCE}}$$

MONOTONE !!! SEQUENCE...

NOW, SINCE f_n <u>NN</u> $\forall n$, IT FOLLOWS THAT

$$\sum_{k=0}^{n+1} f_k = f_0 + f_1 + \ldots + f_n + f_{n+1} =$$

$$= \sum_{k=0}^{n} f_k + \underbrace{f_{n+1}}_{\geq 0}$$

$$\sum_{k=0}^{n} f_k \leq \sum_{k=0}^{n+1} f_k \qquad \forall n$$

$$\sum_{k=0} f_k \leq \sum_{k=0} f_k \quad \forall n$$

APPLY LEVI THM TO MONOTONE SEQUENCE !!!

$$(*) = \sum_{n=0}^{\infty} f_n \overset{DEF}{=} \left(\sum_{k=0}^{n} f_k\right)_{n\in\mathbb{N}} \xrightarrow[n\to\infty]{PW} g$$

SO, SAY

$$g = \sum_{n=0}^{\infty} f_n$$

THE "SUM" (PW LIMIT OF OF THE SEQUENCE OF PARTIAL SUMS)

$$g \overset{PW}{=} \lim_{n\to\infty} \left(\sum_{k=0}^{n} f_k\right)_{n\in\mathbb{N}}$$

MONOTONE SEQUENCE

$$\Downarrow$$

$$\int_E g \overset{THM}{=} \lim_{n\to\infty} \int_E \left(\sum_{k=0}^{n} f_k\right)_{n\in\mathbb{N}} \overset{LINEARITY}{=}$$

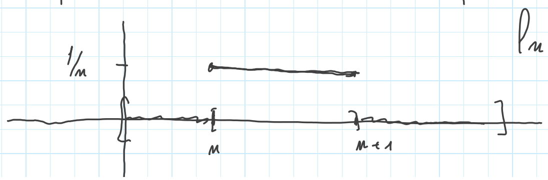
FINITE

IS A SERIES

$$= \overbrace{\lim_{n\to\infty} \left(\sum_{k=0}^{n} \int_E f_k\right)_{n\in\mathbb{N}}}$$

$$\int_E \left(\sum_{n=0}^{\infty} f_n\right) = \sum_{n=0}^{\infty} \int_E f_n \quad !!!$$

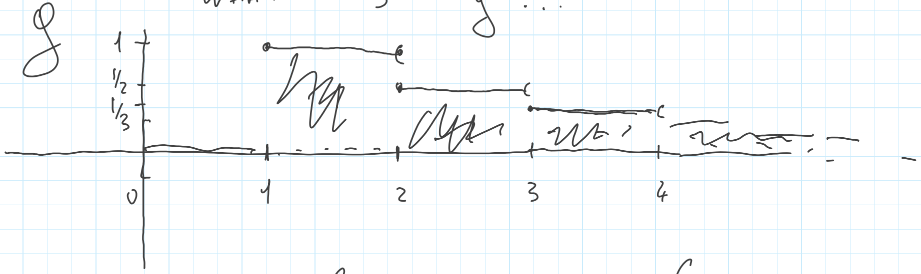
<ins>EX</ins> $f_n : [0,+\infty[\to \mathbb{R}$, $f_n = \frac{1}{n} \cdot \chi_{[n,n+1]} \geq 0$



CONSIDER THE SERIES OF THESE FUNCTS:

$$g = \sum_{n=0}^{\infty} f_n \overset{DEF}{=} \sum_{n=0}^{\infty} \frac{1}{n}\cdot \chi_{[n,n+1[}$$

WHAT IS g ???



WHAT IS $\displaystyle\int_{[0,+\infty[} \left(\sum_{n=0}^{\infty} f_n\right) \overset{DEF}{=} \int_{[0,+\infty[} g$???

WE PROVED

BY THM $\qquad = +\infty$!!!

$$\longrightarrow \int_{[0,+\infty[} \left(\sum_{n=0}^{\infty} f_n\right) = \int_{[0,+\infty[} g \overset{THM}{=}$$

$$= \sum_{n=1}^{\infty} \int_{[0,+\infty[} f_n = \qquad f_n = \frac{1}{n}\chi_{[n,n+1[}$$

$$\overset{DEF}{=} \sum_{n=1}^{\infty} \int_{[0,+\infty[} \frac{1}{n}\cdot \chi_{[n,n+1[} =$$

$$\sum_{n=1}^{\infty} \frac{1}{n}\cdot \mu([n,n+1[) =$$

$$= \sum_{n=1} \frac{1}{n} \int^{n} (\ldots) \qquad \underset{\underline{1}}{\shortparallel}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \quad \leftarrow$$

THE CLASSICAL HARMONIC SERIES !!! ...

DIVERGENT

$$\shortparallel$$

$$+\infty$$

IS IT CLEAR? QUESTIONS?