Wednesday, February 14, 2024 11:36 AM

STEP 3 INTEGRAL OF FUNCTS.

$f : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ f MEAS + f NN.

STEP 4 REMOVE THE ASSUMPTION ~~f NN~~ .

FUNCTIONS $f : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$, f ANY MEASURABLE FUNCTIONS .

SO, GIVEN $f : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ MEASURABLE

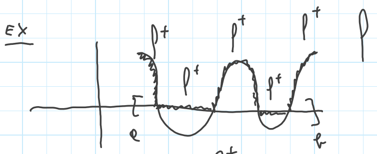
DEFINE :

(*) THE POSITIVE PART f^+ OF f :

$f^+ : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ SUCH THAT

$$f^+(x) = \begin{cases} f(x) & , \text{ IF } f(x) \geq 0 \\ 0 & , \text{ OTHERWISE .} \end{cases} \qquad x \in E$$

EX



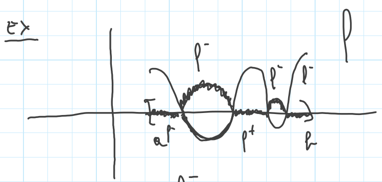
HENCE, f^+ IS NN FUNCTION !!!

(**) THE NEGATIVE PART f^- OF f

IS SUCH THAT

$$f^-(x) = \begin{cases} -f(x) & , \text{ WHEN } f(x) < 0 \\ 0 & , \text{ OTHERWISE} \end{cases}$$

EX



WHAT IS f^- ?

HENCE, f^- IS NN FUNCTIONS !!!

BUT

$$f^+(x) = \begin{cases} f(x) & , f(x) \geq 0 \\ 0 & , \text{ OTHERWISE} \end{cases} \implies$$

$$f^+ = \sup\{0, f\}$$ MAESURABLE + (NN)

MEAS MEAS BY HP

SIMILARLY

$$f^-(x) = \begin{cases} -f(x) & , f(x) > 0 \\ 0 & , \text{OTHERWISE} \end{cases} \Longrightarrow$$

$$f^- = \sup\{\underset{\text{MEAS}}{0}, \underset{\text{MEAS}}{-f}\} \Rightarrow \underline{\text{MEASURABLE}} + (\underline{\text{NN}}).$$

SO, (TRIVIAL) GIVEN $f : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ MEASURABLE

THEN

$$(\dagger) \to \boxed{f = f^+ - f^-} \quad \underline{\text{AND}} \quad |f| = f^+ + f^- . \quad \text{(TRIVIAL)}.$$

LET US "TRY TO DEFINE"

$$(\dagger\dagger) \overset{?}{=} \int_E f \overset{?}{=} \int_E f^+ - \int_E f^- \qquad ???\ldots$$

NOTICE THAT

$$\int_E f^+ , \int_E f^- \qquad \text{EXIST BY STEP 3).}$$

SINCE f^+, f^- $\underline{\text{MEAS + NN}}$

BUT $(\dagger\dagger)$ DOES ALWAY MAKE SENSE?

i) IF $\underline{\int_E f^+ , \int_E f^- < +\infty}$ (f^+, f^- ARE SUMMABLE)

$\underline{\underline{\text{YES}}}$

$$\int_E f = \underset{\in \mathbb{R}}{\int_E f^+} - \underset{\in \mathbb{R}}{\int_E f^-} < +\infty \quad (f \text{ SUMMABLE})$$

ii) IF EITHER

a) $\int_E f^+ < +\infty$, $\int_E f^- = +\infty$

$\underline{\text{YES}}$

$$\int_E f \overset{\text{DEF}}{=} \int_E f^+ - \int_E f^- = \underline{-\infty}$$

$\alpha \in \mathbb{R}$ $\quad +\infty$

OR a') $\int_E f^+ = +\infty \;,\; \int_E f^- < +\infty$

$$\int_E f \overset{DEF}{=} \underbrace{\int_E f^+}_{+\infty} - \underbrace{\int_E f^-}_{\alpha\in\mathbb{R}} = +\infty - \alpha = \underline{+\infty}$$

iii) BUT, IF $\int_E f^+ \;,\; \int_E f^- = +\infty$

$$\int_E f \overset{DEF}{=} \int_E f^+ - \int_E f^- = \infty - \infty$$ THAT MAKES <u>NO SENSE</u>

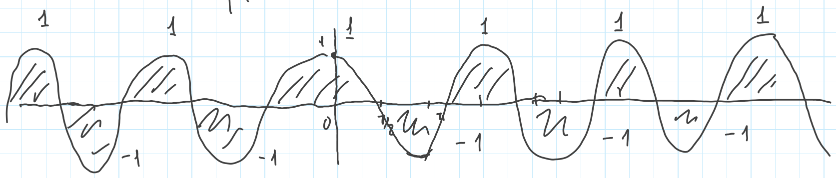
IF f IS SUCH THAT

$\int_E f^+ \;,\; \int_E f^- = +\infty$, THEN

f HAS <u>NO</u> <u>INTEGRAL</u> !!!

<u>NOT SURPRISING</u> CONSIDER $f : \mathbb{R} \to \mathbb{R}$,

$f(x) = \cos x$



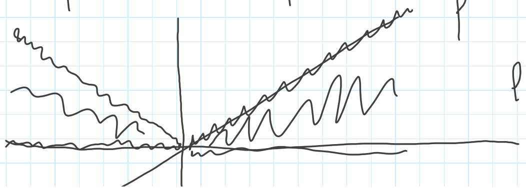
MINUS

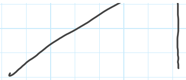
WHAT SHOULD MEAN $\int_{\mathbb{R}} \cos x \overset{?}{=} 1-1+1-1+1-1 \dots$

<u>THIS DOESN'T MAKE SENSE</u>

<u>EX</u> $\quad f : \mathbb{R} \to \mathbb{R} \qquad f(x) = x$

f^- $\qquad$ f $\qquad$ f^+





$$\int_{\mathbb{R}} f^+ = +\infty \quad , \quad \int_{\mathbb{R}} f^- = +\infty$$

THEN $f(x) = x$ IS <u>NOT</u> <u>INTEGRABLE</u> OVER $\underline{\mathbb{R}}$!!!

SO, CONSIDER $f : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ MEAS INTEGRABLE

THAT IS AT LEAST ONE BETWEEN f^+, f^- IS SUMMABLE,

THAT MEAN $\int_E f^+ , \int_E f^- < +\infty$.

SUPPOSE f SUMMABLE $\iff$ f^+, f^- ARE SUMMABLE, THEN

<u>PROP</u>

$c \in \mathbb{R}$

1) $\int_E c \cdot f = c \int_E f$

2) $\int_E (f+g) = \int_E f + \int_E g$

LINEARITY PROP

3) IF $f \leq g$ <u>A.E.</u>, THEN $\int_E f \leq \int_E g$

(MONOTONICITY)

4) $E = A \dot{\cup} B$, $A \cap B = \emptyset$, $A \cup B = E$

THEN

$$\int_E f = \int_A f + \int_B f$$

(ADDITIVITY)

<u>BREAK</u> QUESTIONS?

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