Wednesday, February 14, 2024 11:37 AM

THM (LEBESGUE DOMINATED PW-CONV THM)

LET $g$ MEAS, NN FUNCT $g : E \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$

AND $g$ SUMMABLE $\left( \int_E g < +\infty \right)$

LET $(f_n)_{n \in \mathbb{N}}$ MEAS FUNCTS, $f_n : E \subseteq \overline{\mathbb{R}} \to \mathbb{R}$

SUCH THAT

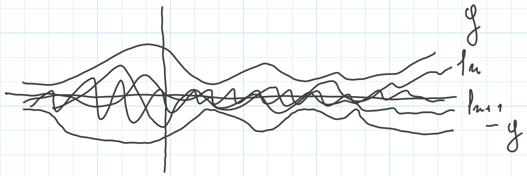
i) $(f_n)$ IS PW-CONV (A.E.) TO A FUNCTION $f : E \subseteq \mathbb{R}^n \to \mathbb{R}$ (MEAS)

$$f_n \xrightarrow[n\to\infty]{PW} f \quad \text{(A.E.)}$$

ii) (DOMINANCE COND.)

$$|f_n| < g \quad \forall n \in \mathbb{N} \quad \Leftarrow \quad |f_n(x)| = g(x) \quad \forall x \in E \quad \forall n \in \mathbb{N}$$

IN PLAIN WORDS




$\int_E g < +\infty$

$\forall n$

THEN

$$\int_E f = \lim_{n\to\infty} \int_E f_n \quad ???$$

NOTICE THAT, $|f_n| < g$ SINCE

$$|f_n| = f_n^+ + f_n^- \Rightarrow$$

$$\int_E |f_n| = \int_E f^+ + \int_E f^- \le \int_E g \overset{HP}{<} +\infty$$

$$\int_E f^+ < +\infty, \int_E f^- < +\infty \Rightarrow$$

$$\int_E f_n = \underset{\substack{\wedge \\ +\infty}}{\int_E f_n^+} - \underset{\substack{\wedge \\ +\infty}}{\int_E f_n^-} < +\infty$$

$f_n$ ARE SUMMABLE $\Rightarrow f \overset{PW}{=} \lim_{n\to\infty} f_n$ IS ALSO SUMMABLE.

PROOF

1) $(g - f_n)$ MEAS + NN

$|f_n| < g$

$f_n \xrightarrow[n\to\infty]{PW} f$

$$\overset{\text{MEAS+NN}}{(g-f_n)} \xrightarrow[n\to\infty]{PW} g-f$$

BY FATOU LEMMA IT FOLLOWS THAT

$$\int_E (g-f) \overset{\text{FATOU LEMMA}}{\le} \min\lim_n \int_E (g-f_n) \Rightarrow$$

$$\int_E g - \int_E f \overset{LIN}{=} \int_E (g-f) \le \min\lim_n \int_E (g-f_n) \overset{DEF}{=}$$

$$\overset{DEF}{=} \sup_n \left(\inf_{k\ge n} \int_E (g-f_k)\right) =$$

LINEARITY

$$\overset{DEF}{=} \sup_n \left(\inf_{k\ge n} \left(\int_E \overset{\text{COST}}{g} - \int_E f_k\right)\right) =$$

$$= \sup_n \left(\int_E \underset{\text{COST}}{g} - \sup_{k\ge n} \int_E f_k\right)$$

$$= \int_E g - \underline{\inf_n \left(\sup_{k\ge n} \int_E f_k\right)} =$$

$$= \int_E g - \max\lim_n \int_E f_n .$$

HENCE, WE PROVED

$$\int_E \overset{\text{COST}}{g} - \int_E f \le \int_E \overset{\text{COST}}{g} - \max\lim_n \int_E f_n$$

$$\Updownarrow$$

$$\rightarrow \int f \ge \max\lim \int f$$

OK ???!!

ii) $\overline{g+f_n}^{\text{MES NN}} \xrightarrow[n\to\infty]{\text{PW}} g+f$ THEN, AGAIN BY FATOU LEMMA

$$\int_E g + \int_E f \overset{\text{LIN}}{=} \int_E (g+f) \overset{\substack{\text{FATOU}\\ \text{LEMMA}}}{\leq} \min\lim_n \int_E (g+f_n) \overset{\text{DEF}}{=}$$

$$= \sup_n \left( \inf_{k\geq n} \left( \int_E (g+f_k) \right) \right) =$$

$$= \sup_n \left( \inf_{k\geq n} \left( \int_E \overset{=\text{COST}}{g} + \int_E f_k \right) \right) =$$

$$= \sup_n \left( \int_E g + \inf_{k\geq n} \left( \int_E f_k \right) \right) =$$

$$= \int_E g + \underbrace{\sup_n \left( \inf_{k\geq n} \left( \int_E f_k \right) \right)}_{\parallel}$$

$$= \int_E g + \min\lim_n \int_E f_n \; .$$

HENCE $\displaystyle \int_E g + \int_E f \leq \int_E g + \min\lim_n \int_E f_n$

$$\Downarrow$$

$$\int_E f \leq \min\lim_n \int_E f_n \qquad \text{OK. !!!}$$

POINT i) $\displaystyle \int_E f \geq \max\lim_n \int_E f_n \qquad (*)$

ii) $\displaystyle \int_E f \leq \min\lim_n \int_E f_n$

BUT $\min\lim \leq \max\lim$ !!! $(**)$

$$\min\lim_{n} \int_E f_n \stackrel{!}{=} \max\lim_{n} \int_E f_n \stackrel{!}{=} \underline{\lim_{n\to\infty} \int_E f = \int_E f} \qquad \underline{\underline{\text{QED}}}$$

<u>RECALL</u> <u>STEP 2</u> FUNCTION $f : E \subseteq \mathbb{R}^n \to \mathbb{R}$

$f$ LIMITED $+ \ \mu(E) < +\infty$

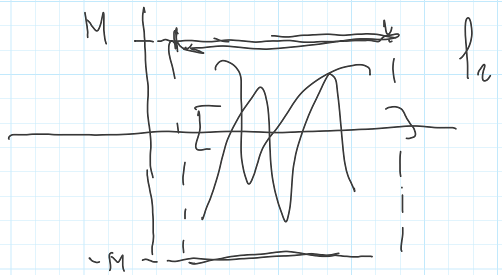
<u>THE (DOMINATED PW CONV) STATES</u>

<u>THM</u> $(f_n)_{n\in\mathbb{N}}$, $f_n$ LIMITED $+ \ \underline{\underline{\mu(E) < +\infty}}$

IF i) $f_n \xrightarrow[n\to\infty]{\text{P.W.}} f$

ii) (DOMINANCE COND.) $\exists\, M \in \mathbb{R}^+$ S.T.

$$|f_n(x)| \le M \qquad \forall x \in E \quad \forall n \in \mathbb{N}$$




THEN $$\int_E f = \lim_{n\to\infty} \int_E f_n \qquad ???$$

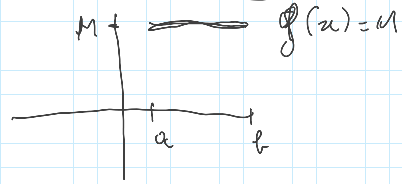
WHY NOW KNOW THE GENERAL

LEBESGUE DOMINATED CONV THM

LEBESGUE DOMINATED CONV. THM

$\Downarrow$

PW DOMINATED CONV. THM OF STEP 2.




INDEED $\exists M \in \mathbb{R}^+$

$$(*) \quad |f_n(x)| \leq M \quad \forall x \in E \quad \forall n \in \mathbb{N}$$

BUT, DEFINE THE FUNCT. $g : E \subseteq \mathbb{R}^n \to \mathbb{R}$

$$g(x) = M.$$

SO COND $(*)$

$$|f_n(x)| \leq M \quad \forall x \in E \quad \forall n \in \mathbb{N} \iff |f_n| \leq g$$

BUT WHAT IS

$$\int_E g = M \cdot \mu(E) < +\infty$$

$\mu(E) < +\infty$

THEN, $g$ SUMMABLE $\Rightarrow$

$\Rightarrow$ LEBESGUE THM APPLIES $\Rightarrow$

$$\Rightarrow \int_E f = \lim_{n \to \infty} \int_E f_n$$

AS DESIDERED

IS IT CLEAR ???