Wednesday, February 14, 2024 11:37 AM

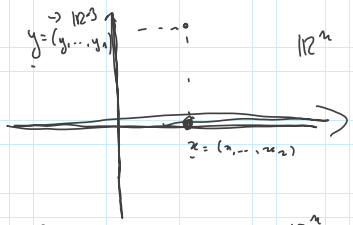
FUBINI / TONELLI THMS (⇒ MULTIPLE INTEGRALS).

PROBLEM LET $B \subseteq \mathbb{R}^n$, B MEASURABLE

$$\mu_n(B) = ?$$

TONELLI THM FOR MEASURES (COROLLARY 5).

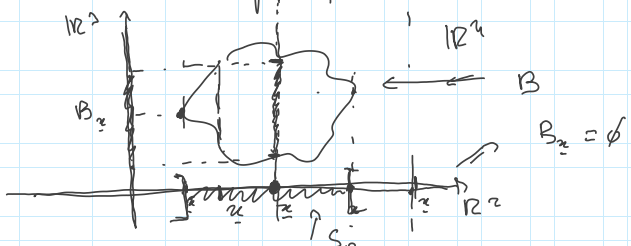
$\mathbb{R}^n = \mathbb{R}^2 \times \mathbb{R}^3$ $\quad$ $(2,3)$ POS. INT. SUCH THAT $2+3=n$



GIVEN A POINT $\underline{x} \in \mathbb{R}^n$:

$$\underline{x} = \big((x_1,\dots,x_2),(y_1,\dots,y_3)\big) = (x,y) \qquad x \in \mathbb{R}^2,\ y \in \mathbb{R}^3$$

CONSIDER $B \subseteq \mathbb{R}^n$, B MEASURABLE



1) $\underline{x} \in \mathbb{R}^2$, $B_{\underline{x}} = \{ y \in \mathbb{R}^3 \,;\, (x,y) \in B \} \subseteq \mathbb{R}^3$

↑ "SECTION" OF B WRT THE FIXED $\underline{x} \in \mathbb{R}^2$

2) $S_B = \{ x \in \mathbb{R}^2 \,;\, \mu_3^*(B_x) > 0 \} \subseteq \mathbb{R}^2$

↑ SUPPORT OF B

3) $S_B^0 = \{ x \in \mathbb{R}^2 \,;\, B_x \text{ IS NOT MEAS.} \} \subseteq S_B \subseteq \mathbb{R}^2$

↑ BAD SUPPORT $\qquad$ ↑ SUPPORT

THEN

1) S_B , S_B^0 ARE MEASURABLE SETS IN $\mathbb{R}^2$

BUT FURTHERMORE $\mu_2(S_B^0) = 0$.

2) $x \in S_B - S_B^0$ ← INDEED A MEASURE !!!

$F: x \longrightarrow \mu_1(B_x)$

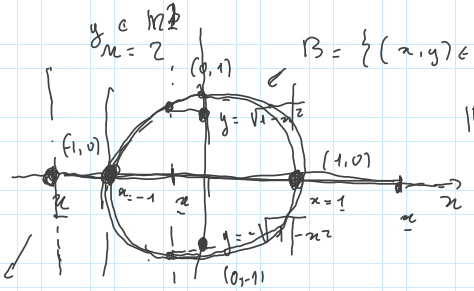
THIS FUNCTION F IS MEAS. + NN.

3) $\mu_n(B) = \int_{S_B - S_B^0} \mu_1(B_x)\, dx$ $\quad \forall x$

$B \subseteq \mathbb{R}^n$

B MEAS

EX 1



$B = \{(x,y) \in \mathbb{R}^2 ;\ x^2 + y^2 = 1\}$ eqn = 1

$\mathbb{R}^2 = \mathbb{R}^1_x \times \mathbb{R}^1_y$

$B_x = \emptyset$ IF $x = -1, 1 \Rightarrow B_x = \{0\}$ $(y = 0)$

$\mu(B_x) = 0$

IF $-1 < x < 1$, $B_x = \{-\sqrt{1-x^2}, \sqrt{1-x^2}\}$

$\Downarrow$

$\mu_1(B_x) = 0$

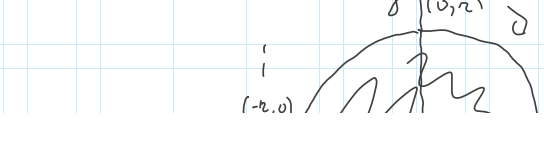
HENCE

$S_B = \{x \in \mathbb{R} ;\ \mu_1^*(B_x) > 0\} = \emptyset$

$\mu_2(B) = \int_{\emptyset} \mu_1(B_x) = 0$!!!

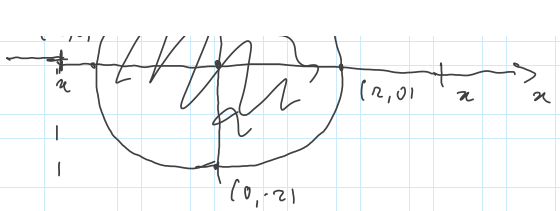
EX 2

$B = \{(x,y) \in \mathbb{R}^2 ;\ x^2 + y^2 \leq 2^2\}$



FULL CIRCLE OF RADIUS 2

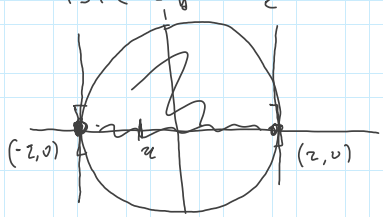
$\mathbb{R}^2 = \mathbb{R}^1 \times \mathbb{R}^1$



i) IF EITHER $x < -2$ OR $x > 2$ THEN

$$B_x = \phi \Rightarrow \mu_1(B_x) = 0$$

ii) LET $|x| \leq 2$



1) IF $x = -2$, $x = +2$

$$B_x = \{0\} \in \mathbb{R}^1_0 \qquad y = 0$$

$$\mu_{\frac{1}{2}}(B_x) = 0$$

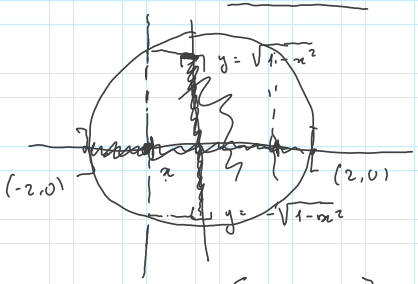
2) IF

$$-2 < x < 2 \Rightarrow B_x = \left[-\sqrt{1-x^2}\,;\sqrt{1-x^2}\right]$$

$$\Downarrow \text{MEAS}$$

$$\mu_1(B) = 2\sqrt{1-x^2} > 0$$

$|x| \leq 2$



SO $S_B =]-2,2[$ AND $S_B^0 = \phi$.

SO

$$\mu_2(B) = \int_{S_B \setminus S_B^0} \mu_1(B_x)\,dx =$$

$$= \int_{]-2,2[} \left(2\sqrt{1-x^2}\right) dx$$

$$\int_{-2}^{2}$$

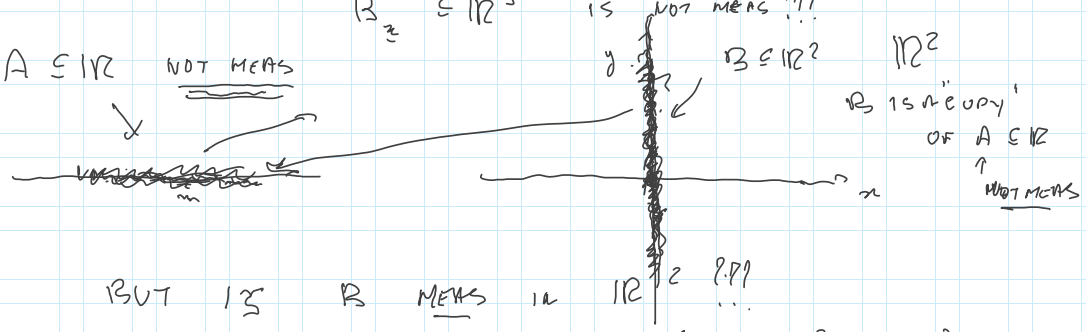
$$= R \int_{-2}^{2} \left(2\sqrt{1-x^2}\right) dx = \underline{\underline{\pi 2^2}}$$

IS IT CLEAR ???

WHY, GIVEN $B \subseteq \mathbb{R}^n$ MEASURABLE,

IT MAY HAPPEN FOR SOME $x \in \mathbb{R}^2$ THAT

$B_x \subseteq \mathbb{R}^1$ IS NOT MEAS ???

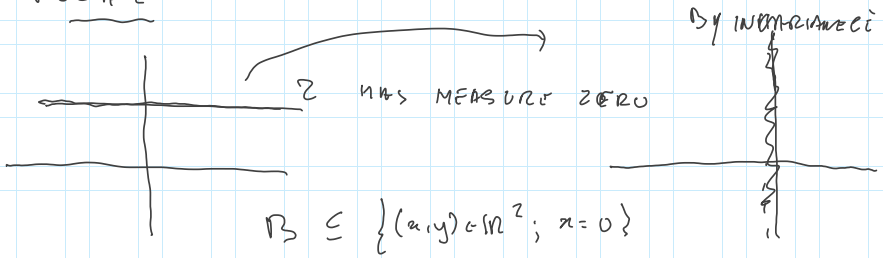


BUT IF B MEAS IN $\mathbb{R}^2$???

NOTICE THAT, IN $\mathbb{R}^2$, $B \subseteq \{(x,y) \in \mathbb{R}^2 ; x = 0\}$

A LINE IN $\mathbb{R}^2$

RECALL



$$B \subseteq \{(x,y) \in \mathbb{R}^2 ; x = 0\}$$

$$0 = \mu_2^*(B) \leq \mu_2\left(\{(x,y) \in \mathbb{R}^2 ; x = 0\}\right) = 0$$

SO $\mu_2^*(B) = 0 \Rightarrow$ B MEAS IN $\mathbb{R}^2$

LET $x = 0$, WHAT

$$B_x = \{y \in \mathbb{R} ; (0,y) \in B\} \subseteq \mathbb{R}^1$$