Riemannian geometry

Notes

(*) means that the proof of the theorem is not required for the exam(**) means that the theorem/the argument is not required for the exam (***) means that the argument is optional.

Contents

1. Topological manifolds

Informally we can think that manifolds are generalization of surfaces and curves to higher dimensions. The dimension of the manifolds would be the number independent parameters needed to specify a "point"; thus an n dimensional manifold is in some sense an object modelled locally on $\mathbb{R}^n$. The easiest example we can visualize is the sphere S^2 given by the equation $x^2 + y^2 + z^2 = 1$; near the north pole $(0, 0, 1)$ we can solve for z as a function of x and y thus we would need two parameter to specify a point close to the north pole and that's why we will say that the sphere is a 2 dimensional manifold in the sense the "locally" looks like $\mathbb{R}^2$. But what do we mean with *looks like*??? the main idea is that $U \in \mathbb{R}^k$ and $V \in \mathbb{R}^n$ are said to be homeomorphic if we can find a one to one correspondence $\varphi : U \to V$ s.t. φ and its inverse are continuous map.

Let's try to be a bit more "mathy" and abstract now. We want to come up with a notion of "space" in which the notion continuous function makes sense. Let's discuss the case of $\mathbb{R}^n$ for simplicity the generalization to any metric space is obvious. A set U in $\mathbb{R}^n$ is said to be open if for every p in U there is an open ball $B_r(p)$ such that $B_r(p) \subset U$. A neighborhood of p is an open set containing p. It is clear that the union of an arbitrary collection of open sets is open, but NOTE the same is not true of the intersection of infinitely many open sets. A map f from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^m$ is continuous if and only if the inverse image $f^{-1}(V)$ of any open set V in $\mathbb{R}^m$ is open in $\mathbb{R}^n$. This shows that continuity can be defined in terms of open sets only for $\mathbb{R}^n$ as for every metric spaces [1]. More in details first one defines open ball $B_r(p) = \{x \in \mathbb{R}^n \ s.t. \ d(x,p) < r\}$ where $d(x,y)$ is the usual euclidean distance. We define axiomatically thus open sets looking at what happens in $\mathbb{R}^n$

Definition 1.1. Let X be a set; a *topology* on X is a collection τ of subsets of X, called open sets,such that

- $X, \emptyset \in \tau$
- $U_1, .., U_n \in \tau \to U_1 \cap .. \cap U_n \in \tau$ (intersection of finite number of opens)
- $\{U_i\}$ finite or infinite collection of elements of $\tau \to \bigcup U_i \in \tau$

We call the pair (X, τ) or simply X, a *topological space*

The idea of open sets it is somehow needed to recover the notion of nearness we would have in a metric space. In this sense a neighborhood of a point $p \in X$ is just an open set U containing p. A natural example of topology of a (finite) set S is the collection of all subsets of S. This is sometimes called the discrete topology.
Let's resume an example: Consider the set $S = \{1, 2, 3\}$

- $\tau = \{\{1\}, \{1, 2\}, \{1, 2, 3\}, \{\emptyset\}\}$ is a topology
- $\tau = \{\{S\}, \{\emptyset\}\}$ is the trivial topology
- $\tau = \{\{1\}, \{2\}, \{3\}\{1, 2\}, \{1, 3\}, \{2, 3\}\{1, 2, 3\}, \{\emptyset\}\}$ is the discrete topology

Once we have a top. space (S, τ) a subset A naturally have a top. space structure (A, τ_A) given by

$$\tau_A := \{U \cap A \ with \ U \in \tau\}$$

This is sometimes called subspace topology or relative topology.
Take now a topological space X and a set S and $\pi : X \to S$ a surjective map. We can naturally define a topology on S by saying that $U \in S$ is open if $\pi^{-1}(U)$ is open. This is called *quotient topology.* In general it's hard to describe all the open sets in τ and we introduce thus the notion of a base.

Definition 1.2. A subcollection β of a topology τ of a set S is a basis for τ if for every $U \in \tau$ and $p \in U$ we can find $V \in \beta$ s.t. $p \in V \subset U$

An useful criterion for deciding if a collection of subsets is a basis for some topology is given by the following:

Proposition 1.3. *A collection $\beta = \{u_i\}$ of subset of S is a basis for some topology of S if and only if*

- *S is the union of all u_i*
- *given u_i and u_j and $p \in u_i \cap u_j$ there is $u_k \in \beta$ s.t. $p \in u_k \subset u_i \cap u_j$*

Another point of view or if you prefer a consequence of the previous definition is to say that β is a basis if every open set of S can be recovered by a union of sets in β.

Definition 1.4. A top space is named second countable if it admits a countable basis.

Sometimes one may want that open sets separate points. More precisely

Definition 1.5. A topological space S is Hausdorff if given any two distinct points x and y in S, there exist disjoint open sets U, V such that $x \in U$ and $y \in V$.

[1] For metric space we say that given (X, d_X) and (Y, d_Y) be metric spaces a function $f : X \to Y$ is continuous at $p \in X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $d_X(x,p) < \delta$ implies that $d_Y(f(x), f(a)) < \epsilon$.

We now go back to the concept of continuity of a function. Let $f : X_1 \to X_2$ be a function of topological spaces. Mimicking the definition from advanced calculus, we say that f is continuous at a point p in X_1 if for every neighborhood U_2 of $f(p)$ in X_2 (that is an open set containing $f(p)$, there is a neighborhood U_1 of p such that $f(U_1) \subset U_2$. In the case of quotient topology one can prove that a function from the quotient space is continuous IFF its composition with the quotient map is continuous.

The notion of equal in the "category" of top spaces is encoded in the definition of homeomorphism

Definition 1.6. Two top space X and Y are said to be homeomorphic or topologically equivalent if exists an homeomorphism between them that is a cont and bijective map with cont inverse.

Definition 1.7. A topological space is disconnected if it is the union of 2 disjoint non empty open sets. If it is not disconnected we will call it connected.

Definition 1.8. An *open cover* of a top space S is a collection of open subsets whose union is S. A subcover is a subcollection still covering S. We will say that S is *compact* if every open cover admits a finite subcover.

Fact: $\mathbb{R}^n$ is second countable and and subset of a second countable top space is second countable. One of the most important property of a second countable top space is that any open cover admits a countable subcover.

A subset A of a top space S is named closed if its complement is open. Note that closed does not mean not open for example given the top space S the open set S is also closed being its complement the empty set. We can define the closure of a set A in S by

$$\bar{A} = \cap\{B \subset S \,:\, A \subset B \,\text{and}\, B \,\text{is closed}\}$$

It is time to translate to the math language the notion of "looks like $\mathbb{R}^n$". A topological space M is said to be locally Euclidean of dimension n if every point $p \in M$ has a neighborhood U that is homeomorphic to an open subset of $\mathbb{R}^n$, that is we have an homeomorphism $\varphi : U \to \widetilde{U} \subset \mathbb{R}^n$. Let's call the pair (U, φ) the local *coordinate chart* and we sometimes refer to U as coordinate open set. If the image of φ is an open ball of $\mathbb{R}^n$ we will call U Euclidean open ball in M if $\varphi(p) = 0$ we say that the chart is centered in p.

Definition 1.9. A subset of a top space is called precompact if its closure is compact.

Definition 1.10. A topological manifold is a top space that is Hausdorff second countable and locally Euclidean.

Note that open subset of a top manifold is a top manifold. Let's discuss now a couple of examples:

- S^n is for free Hausdorff and second countable bc it lives inside $\mathbb{R}^{n+1}$ and we can use the subset topology. Thus we need just to check that it's locally Euclidean. To this aim consider:

$$U_i^{\pm} = \{\mathbf{x} = (x_1, .., x_{n+1}) \in S^n \;s.t.\; \pm\, x_i > 0\}$$

equipped with the following homomorphism

$$\begin{array}{rrcl} \varphi_i^{\pm}: & U_i^{\pm} & \to & \widetilde{U}_i \subset \mathbb{R}^n \\ & (x_1,..,x_{n+1}) & \mapsto & (x_1,..,x_{i-1},x_{i+1},..,x_{n+1}) \end{array}$$

with continuous inverse

$$\begin{array}{rrcl} (\varphi_i^{\pm})^{-1}: & \widetilde{U}_i \subset \mathbb{R}^n & \to & U_i^{\pm} \\ & (z_1,..,z_n) & \mapsto & (z_1,..,z_{i-1},\pm\sqrt{1-|z|^2},z_{i+1},..,z_n) \end{array}$$

This choice is often call *graph coordinate chart*. Another opportunity to construct a coordinate chart for the sphere is to use the stereographic projection.

- $\mathbb{RP}^n$: this is the set of lines in $\mathbb{R}^{n+1}$. Mathematically we can define it by the map $\mathbb{R}^{n+1} \setminus \mathbf{0} \to \mathbb{RP}^n$ sending each point into the line passing trhough itself and the origin. Note that for any $\lambda \in \mathbb{R} \setminus 0$ we have $\pi(\mathbf{x}) = \pi(\lambda\mathbf{x})$ thus the projective space can be also defined by dividing $\mathbb{R}^{n+1} \setminus \mathbf{0}$ by the equivalence class $\mathbf{x} \sim \lambda\mathbf{x}$. Now we equip $\mathbb{RP}^n$ with the quotient topology that one can easily check that is Hausdorff. We want to prove now that it second countable and locally Euclidean. Let's start with the last one. Define $V_i \in \mathbb{R}^{n+1} \setminus \mathbf{0}$ to be the set where the i-th component of $\mathbf{x}$ is non vanishing and define $U_i = \pi(V_i)$.Then consider

$$\begin{array}{rrcl} \varphi_i: & U_i^{\pm} & \to & \widetilde{U}_i \subset \mathbb{R}^n \\ & [(x_1,..,x_{n+1})] & \mapsto & (x_1/x_i,..,x_{i-1}/x_i,x_{i+1},..,x_{n+1}/x_i) \end{array}$$

with inverse

$$\begin{array}{rrcl} (\varphi_i)^{-1}: & \widetilde{U}_i & \to & U \\ & (z_1,..,z_n) & \mapsto & [(z_1,..,z_{i-1},1,z_{i+1},..,z_n)] \end{array}$$

This choice is often call *canonical coordinate chart*. We may want these maps to be continuous, well in the first case it is enough to note that $\varphi_i \circ \pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is continuous while for the inverse it is easy to prove that open goes to open.
Is it enough to declare $\mathbb{RP}^n$ a topological manifold?? Yes and no..because we need second countability but it is somehow for free thank to the following Lemma (see for example Lee, Introduction to topological manifolds, Lemma 3.2.1):

Lemma 1.11. Suppose $M \to N$ is a quotient map and M second countable (for example a top. manifold) then if N is locally Euclidean then is also second countable. *Invoking this Lemma we have all the properties we need to prove that the projective space is a topological manifold.*

2. Smooth manifolds

We now want to add more structure to our manifolds in order to define functions derivative of functions or more in general the concept of smoothness, in a consistent way. In particular if we consider a topological manifold M we are tempted to define a function $f : M \to \mathbb{R}$ smooth at $p \in M$ if $\hat{f}_\varphi := f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}$ is smooth at $\hat{p} := \varphi(p)$ for some coordinate chart (U, φ). The problem of this definition is that it depends too much on the choice of the coordinate chart. Consider for example two charts (U, φ_U), (V, φ_V) with $U \cap V \neq \emptyset$ and a point $p \in U \cap V$ then we may obtain that in the coordinate chart U

$$\hat{f}_{\varphi_U}|_U := f \circ \varphi_U^{-1}|_{\varphi(U)}$$

is smooth at $\varphi_U(p)$ but this does not imply that is also smooth "in the other chart" that is $\hat{f}_{\varphi_V}$ is smooth at $\varphi_V(p)$. In the intersection $U \cap V$ in the V chart we have

$$\hat{f}_{\varphi_V}|_{U\cap V} := f\circ\varphi_V^{-1}|_{\varphi_V(U\cap V)} = f\circ\varphi_U^{-1}\circ(\varphi_U\circ\varphi_V^{-1})|_{\varphi_V(U\cap V)} = \hat{f}_{\varphi_U}|_{\varphi_U(U\cap V)}\circ(\varphi_U\circ\varphi_V^{-1})|_{\varphi_V(U\cap V)}$$

Then if $\varphi_U \circ \varphi_V^{-1}$ is not good enough we can find a contradictions, that is f is smooth in a given coordinate chart but not in the other. Let's solve this problem. Let's consider two chart (U_i, φ_i) and (U_j, φ_j) and let's define $U_{ij} := U_i \cap U_j$; we then construct the so called *transition function* $\varphi_{ij} := \varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_{ij})} : \mathbb{R}^n \supset \varphi_i(U_{ij}) \to \varphi_j(U_{ij}) \subset \mathbb{R}^n$ (we will often omit the domain of the transition function to simplify the notation). In the case of topological manifolds the transition functions are obviously homomorphisms. But now we want something more.

Notation: Once we choose a coordinate chart (U, φ) it is convenient to ease the notation. We will mostly in fact use the following notation $\tilde{U} = \varphi(U)$ and $\hat{f} = f \circ \varphi^{-1}|_{\varphi(U)} : \tilde{U} \to \mathbb{R}$, it will be clear for the context what we are doing.

Let's remember to ourselves the following definition:

Definition 2.1. A smooth map from an open set V of $\mathbb{R}^n$ to an open set W of $\mathbb{R}^m$ bijective and with a smooth inverse is called a diffeomorphism

That's what we need.

Definition 2.2. Two coordinate charts (U_i, φ_i) and (U_j, φ_j) are named smoothly compatible if $U_{ij} = \emptyset$ or φ_{ij} is a diffeo.

Now we put all charts together in a clever way.

Definition 2.3. A smooth atlas, or simply an atlas, A, for a topological manifold M is a collection of smoothly compatible charts covering M.

Note that it is enough to check that every transition function φ_{ij} is smooth (thus for all i, j) to prove that we have an Atlas. We thus successfully removed the ambiguity of the definition of a smooth function on M once we fixed the Atlas; a function $f : M \to \mathbb{R}$ then is smooth if $\hat{f} = f \circ \varphi^{-1}$ is smooth for every coordinate chart in the atlas A. We are tempted to call this a "smooth structure". A minor problem appears now. Consider for example on $\mathbb{R}^2$ the following atlases

$$A_1 := \{(\mathbb{R}^2, id)\}$$

and

$$A_2 := \{(B_1(\mathbf{x}), id)\}$$

they are obviously different but they determine the same set of smooth functions. The way out is easy

Definition 2.4. A smooth structure is an equivalence class of atlases where we will declare the atlas A_1 equivalent to A_2 if $A_1 \cup A_2$ ia another smooth atlas

Alternatively one can say the following

Definition 2.5. A smooth structure is a maximal Atlas A^{max}, one thus that is not contained in a larger atlas.

Lemma 2.6. *(1) Every smooth atlas A for M is contained in a unique max atlas*
(2) Two atlases determine the same max atlas IFF their union is an atlas

Proof. Define now A^{max} to be the set of all possible chart compatible with those in A; note that this means that is maximal by construction. Is it an Atlas? A^{max} obviously cover M thus we need to prove compatibility of the charts. Lets consider (V_1, ψ_1) and (V_2, ψ_2) charts in A^{max} ans suppose $V_1 \cap V_2 \neq \emptyset$; are the compatible? that is $\psi_1 \circ \psi_2^{-1}$ is a diffeo? Take a point $p \in V_1 \cap V_2$, then we can find a chart (U, φ) in A with $p \in U$. Observe now that

$$\psi_1 \circ \psi_2^{-1} = \psi_1 \circ (\underbrace{\varphi^{-1} \circ \varphi}_{\text{we insert them here}}) \circ \psi_2 = \underbrace{(\psi_1 \circ \varphi^{-1})}_{\text{smooth}} \circ \underbrace{(\varphi \circ \psi_2)}_{\text{smooth}}$$

Thus $\psi_1 \circ \psi_2^{-1}$ is smooth in a neighborhood of p or better to say $\psi_2(p)$, thus it is smooth in $\psi_2(V_1 \cap V_2)$. Obviously also the inverse is smooth thus this transition function is a diffeo and A^{max} an Atals. Uniqueness is easy to prove, consider B^{max} another maximal atlas for A, since every chart of B^{max} are compatible with the chart in A we obviously get that $B^{max} \subseteq A^{max}$, but since we claim that B^{max} is maximal then we have $B^{max} = A^{max}$.

Let's see now the second part

$(\rightarrow)$ obvious

$(\leftarrow)$ Let $A_1 = (U_i, \varphi_i)$ and $A_2 = (V_a, \psi_a)$. $A_1 \cup A_2$ atlas implies that all the chart on A_1 are also compatible with the one one A_2 thus we have that $\varphi_i \circ \psi_a$ is a diffeo. Using that info, and after having constructed $A_1^{max} := (U_I, \phi_I)$ and $A_2^{max} := (V_A, \Xi_A)$ we observe that (we are sloppy with the domain and image of those functions in order not to be pedantic)

$$\varphi_i \circ \psi_a^{-1} = \text{smooth} = \varphi_i \circ (\phi_I^{-1} \circ \phi_I) \circ \psi_a^{-1} = \underbrace{(\varphi_i \circ \phi_I^{-1})}_{\text{smooth}} \circ \phi_I \circ \psi_a^{-1}$$

then we have that $\phi_I \circ \psi_a^{-1}$ must be smooth. This implies that $A_1^{max} \subseteq A_2^{max}$. In a similar fashion, or if you prefer by democracy, one also has $A_2^{max} \subseteq A_1^{max}$ then the statement follows. □

We are finally ready to define a smooth structure

Definition 2.7. A smooth manifold is the information of a topological mfd and a smooth structure

Definition 2.8. Let M be a smooth manifold. A function $f : M \to \mathbb{R}$ is said to be smooth if $\forall (U, \varphi) \in A^{max}$ we have that $f \circ \varphi^{-1}|_{\varphi(U)}$ is smooth

You may think that we have a problem bc we can't check for all charts on the max atlas..they are too many. No problem:

Lemma 2.9. *Given a smooth manifold M with smooth structure given by the max atlas A^{max}, then fixed an atlas $A = \{(u_i, \varphi_i)\} \subset A^{max}$ then $f : M \to \mathbb{R}$ is smooth (in the sense of the previous def) if $f \circ \varphi_i^{-1}|_{\varphi_i(U)}$ is smooth for all i.*

Examples

- Take $\mathbb{R}^n$ with the standard chart $(\mathbb{R}^n, id)$, this is obviously a smooth manifold.
- Let U be any open subset of $\mathbb{R}^n$. Then U is a topological manifold, and the single chart (U, Id) defines a smooth structure on U. This is in general true for every open set of a smooth manifold.

- Any real finite dimensional vector space V. Take any positive definite scalar product on V in order to induce on V the structure of a topological mfd (Hausdorf and second countable). Let's discuss the smooth structure. We know that any choice of a basis $\mathbf{B}$ establish an isomorphism between V and $\mathbb{R}^n$ as follows
$$\varphi_{\mathbf{B}} : \mathbf{v} \to \mathbf{v_B} = (v_1, .., v_n) \,\text{coordinate in the given basis } \mathbf{B}$$
and $(V, \varphi_{\mathbf{B}})$ is our smooth structure, since in particular $\varphi_{\mathbf{B}}$ is an hoeomorphism. Suppose now we want to use another basis $\tilde{\mathbf{B}}$ where $\mathbf{v}_{\tilde{\mathbf{B}}} = (\tilde{v}_1, .., \tilde{v}_n)$, we know we can always find an invertible matrix A_i^j so that
$$\tilde{v}_i = A_i^j v_j$$
thus in general an invertible linear map
$$\begin{array}{rccc} \varphi_{\mathbf{B}\tilde{\mathbf{B}}} & : \mathbb{R}^n & \to & \mathbb{R}^n \\ & \mathbf{v_B} & \to & \mathbf{v}_{\tilde{\mathbf{B}}} \end{array}$$
that is our construction does not depend on the choice of the basis as expected in the sense that the two chart are compatible and thus the smooth structure induced is the same.
- Any open subset U of $\mathbb{R}^n$ is a topological manifold and the restriction of every single chart of the given atlas for $\mathbb{R}^n$ defines a smooth structure on U. this generalize to any open subset of a smooth manifold.
- The set of real invertible matrices $GL_n(\mathbb{R})$ can be defined by as $\det^{-1}(\mathbb{R} \setminus 0)$ where
$$\det : \mathbb{R}^{n^2} \to \mathbb{R}$$
where we have identified square matrices with $\mathbb{R}^{n^2}$. Being the determinant a continuous function and $(\mathbb{R} \setminus 0$ an open set we can conclude that $GL_n(\mathbb{R})$ is open inside $\mathbb{R}^{n^2}$ and thus a smooth manifold
- More example in the exercise sheet 1

Now we want generalize a bit the construction and we look at maps between smooth manifolds M and N, $F : M \to N$. We consider the smooth atlases (U_i, φ_i) and (V_a, ψ_a); we say thaty F is smooth if $\psi_a \circ F \circ \varphi_i^{-1} : \mathbb{R}^n \supset \varphi_i(U_i \cap F^{-1}(V_a)) \to \psi_a(V_a) \subset \mathbb{R}^m$ for every (U_i, φ_i) and (V_a, ψ_a). Sometimes we we just write $\hat{F}$ for the coordinate representation of F in some coordinate charts.

Definition 2.10. A map between smooth mfds $F : M \to N$ is a diffeomorphism if it is smooth with a smooth inverse. In this case we would say that M and N are diffeomorphic

Note that the notion of being diffeomorphic depends on the smooth structure chosen We have discuss previously that one can equip a top manifold with different atlases. In case they are compatible we thus work with the same smooth structure, but in case not we are temped to say that the manifolds are different from the "smooth" point of view. This sentence is probably too strong. In the end in fact we may add different smooth structures to a top manifold but end up with diffeomorphic smooth mfds. Let s see it with an example. Consider $\mathbb{R}$ with the standard atlas $\varphi = id$ or with $\psi(x) = x^3$ and denote for simplicity $\mathbb{R}_\varphi$ and $\mathbb{R}_\psi$ the smooth mfds arising form those smooth structures. They determine different smooth structures on $\mathbb{R}$ in fact the transition function $\varphi \circ \psi^{-1}(x) = x^{\frac{1}{3}}$ that is not smooth at the origin. Anyhow defining $F : \mathbb{R}_\varphi \to \mathbb{R}_\psi$ $F(x) = x^{\frac{1}{3}}$ we can conclude that they are diffeomorphic since $\hat{F} = id$. For this point of view the right quastion one could try to ask is how many smooth structure up to diffeomorphism one can add to a top manifold.

3. The partition of unity

In the context of topological spaces it is often usefull the gluing Lemma that is need to "glue" continuous functions defined on open or closed subsets. In the case of smooth manifold we have something weaker:

Lemma 3.1. *Take M and N smooth mfds and $\{U_i\}$ an open cover of M. Suppose then that we have a collection of smooth functions $F_i : U_i \to N$ agreeing on the overlap then exists a unique smooth map $F : M \to N$ such that $F|_{U_i} = F_i$.*

What about closed subsets?? Take as a counterexample $M = N = \mathbb{R}$ with the standard smooth structure, $U_- = [-1, 0]$ and $U_+ = [0, 1]$ and $f_\pm = \pm x$. They agree on the overlap obviously but their "union" $|x|$ is not smooth at the origin.

Question: is there a way to have a weaker version of the gluing Lemma for smooth manifolds, namely blend together local smooth objects without assuming that they agree on "too big" overlaps ??
Answer: Yes it is called the partition of unity, and it is a crucial tool in differential geometry. The idea is to construct functions that are identically vanishing in specified parts of a manifolds. A partition of unity is used in two ways:

- to decompose a global object on a manifold into a locally finite sum of local objects on the open sets $\{U_i\}$ of an open cover
- to patch together local objects on the open sets $\{U_i\}$ into a global object on the manifold.

Thus, a partition of unity serves as a bridge between global and local analysis on a manifold. This is useful because while there are always local coordinates on a manifold, there may be no global coordinates. It is the single feature that makes the behaviour of smooth manifolds so different from that of real-analytic or complex manifolds.

One of the main tool needed will be the bumb function that we will now introduce. Let us briefly remind ourselves that given a smooth function $f : M \to \mathbb{R}$ on M, we can define its support as follows:

$$\text{supp}(f) = \text{closure}\{p \in M \, s.t. \, f(p) \neq 0\}$$

and we say f supported U if $\text{supp}(f) \subseteq U$

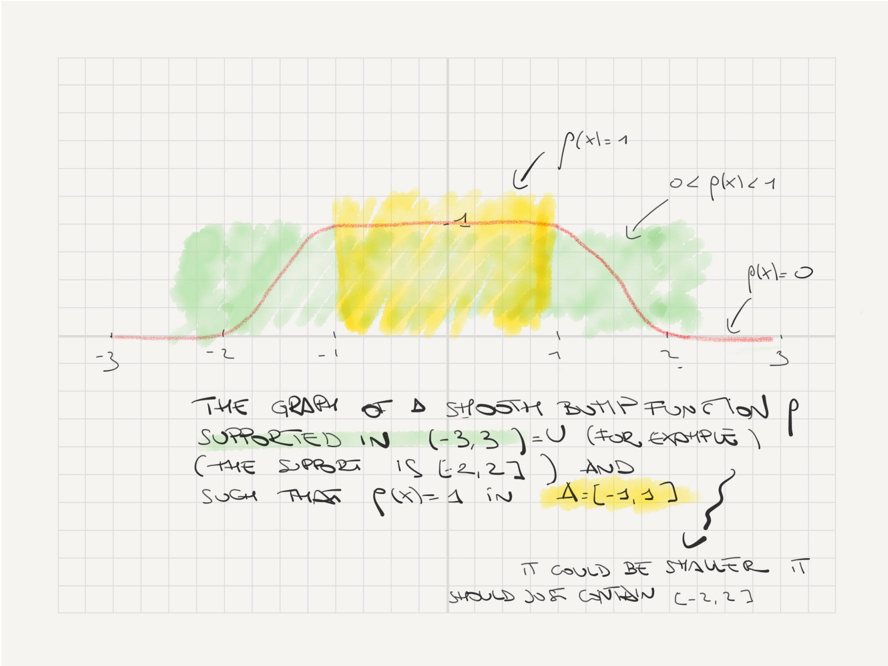
Definition 3.2. A smooth function $\beta : \mathbb{R}^n \to \mathbb{R}$ is called a BUMP function at p supported in $U \subset \mathbb{R}^n$ if it is equal to one in a neighborhood A of p and $\text{supp}(\beta) \subset U$.

We now construct an example of useful and non trivial bump function.
Fact: The function

$$f(t) := \begin{cases} e^{-\frac{1}{t}} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

This is called *cut-off function*, it is smooth (BUT not analytic). Moreover the following function we can construct out of the previous one

$$h(t) := \frac{f(2-t)}{f(2-t) + f(t-1)}$$



that is such that $h(t) = 1$ when $t \leq 1$, $0 < h(t) < 1$ when $1 < t < 2$ and zero otherwise. Note that this can be generalized taking every real numbers r and R with $r < R$ and considering

$$h(t) = \frac{f(R-t)}{f(R-t) + f(r-1)}$$

but in the following we will consider for simplicity the case $r = 1$ and $R = 2$.

Fact: The function $\rho(\mathbf{x}) : \mathbb{R}^n \to [0, 1]$ defined by $\rho(\mathbf{x}) = h(|\mathbf{x}|)$ is a bump function at the origin supported in any open set containing $\bar{B}_2(\mathbf{0})$ and identically equal to 1 in $B_1(\mathbf{0})$. We call the function $\rho(\mathbf{x} - q)$ a bump function at $q \in \mathbb{R}^n$. First of all we observe that this construction can be generalized to a general smooth manifold.

Theorem 3.3. *Let M be a smooth manifold. For every point $p \in M$, and every neighborhood V of p, there exists a smooth function β such that $\beta = 1$ in a neighborhood A of p and the support of β is contained in V, that is a bump function at $p \in A$ supported in V.*

Proof. Take a coordinate chart (U, φ) centred at p (this is always possible) with $U \subset V$ and define for every $q \in U$, with an abuse of notation, the function $\beta(x) = \rho \circ \varphi(q)$ and extend then it consistently to vanish outside U . This construction assume that $\varphi(U)$ contains the ball of radius 2, otherwise we can rescale the balls of radius 1 and 2 accordingly as suggested at the beginning of this section. Moreover by construction we have that sup$\beta = \varphi^{-1}(\bar{B}_2(\mathbf{0})) \subseteq U \subseteq V$. Since φ is an homeomorphism (and thus preserves compactness) we have that supβ is a compact subspace (a subset with the subspace topology) and since M is Hausdorf is also a closed subspace (see proposition 4.19 of the book Lee, Introduction to topological manifolds). With an abuse of notation we call the function β constructed above ρ as in the $\mathbb{R}^n$ case. $\square$

The next goal is to use bump function efficiently. For example:

Proposition 3.4. *Suppose f_U is a smooth function defined on a neighbourhood U of q in a manifold M. Then there is a smooth function f on M which agrees with f_U in some possibly smaller (w.r.t. U) neighbourhood of q.*

Proof. It is enough to take a bump function at q supported in U identically 1 in a neighbourhood of p and then define:

$$f := \begin{cases} \rho(x)f(x) & \text{in } U \\ 0 & \text{otherwise} \end{cases}$$

. □

We want something more, we want to "glue" smooth functions possibly in disagreement on overlaps.

Definition 3.5. We say that a collection of subsets $\{U_i\}$ of a top space is locally finite if each point admit a neighbourhood that intersect at most finitely many of U_i

Definition 3.6. Consider $\mathfrak{U} = \{U_i\}_{i\in I}$ an open cover of a smooth manifold M. A partition of unity subordinate to $\mathfrak{U}$ is a collection of smooth functions $\phi_i : M \to \mathbb{R}$ (they are not coordinate charts !!!) such that

- $0 \le \phi_i(p) \le 1$
- $\text{supp}(\phi_i) \subset U_i$
- the set $\{\text{supp}\,\phi_i\}$ is locally finite (this is needed to make sense to 1) when I is a infinite set)
- $\sum \phi_i(p) = 1 \ \forall p \in M$

Remark Suppose now $\{f_i\}$ is a collection of smooth functions on a manifold M such that $\{\text{supp}(f_i)\}$ is locally finite. Then every point p in M has a neighbourhood U_p that intersects finitely many $\text{supp}(f_i)$; thus $\sum_i f_i$ is actually a finite sum on U_p and thus the function $\sum f_i$ is well defined and smooth on the whole manifold M. It is a natural question whether a smooth manifold admits a partition of unity. Let's start with something somehow simpler (this simplified version comes from the book of Loring W. Tu, Introduction to manifolds, section 13.3)

Proposition 3.7. *Let M be a compact manifold and $\mathfrak{C} = \{C_a\}_{a\in A}$ an open cover. There exists a partition of unity subordinate to $\mathfrak{C}$*

Proof. First of all we note, without proving, that, given $f_1, .., f_n$ functions on a manifold M then

$$\text{supp}(\sum f_i) \subseteq \bigcup_i \{\text{supp}(f_i)\}$$

Now for each $p \in M$ we define with ρ_p the bumb function for $p \in C_a$ supported in C_a. We call A the set of all possible values of a. We call now W_p a neighbourhood of p where $\rho_p > 0$. Now the crucial point is that being M compact by assumption the cover $\{W_p, p \in M\}$ has a finite subcover we name $\{W_{p_1}, .., W_{p_m}\}$. Call ρ_{p_i} the corresponding bumb functions as constructed before and construct $\psi = \sum_{i=1}^m \rho_{p_i}$. Note that since every $p \in M$ belong to some W_{p_i} then ψ never vanishes. Define then

$$\varphi_i := \frac{\rho_{p_i}}{\psi}$$

Note that $\sum \varphi_i = 1$ and $\text{supp}\varphi_i \subset C_a$ for SOME $a \in A$. This is really close to a partition of unity, we need just to reindex to get the given open cover as part of the game. Call

$\alpha(i) \in A$ to be the index so that

$$\text{supp}\varphi_i \subset C_{\alpha(i)}$$

and fixed $a \in A$ call $\rho_a := \sum_{i\,\text{s.t.}\,\alpha(i)=a} \varphi_i$. If there is no i s.t. $\alpha(i) = a$ we set ρ_a to zero. It is easy to check that $\{\rho_a\}$ is a partition of unity for C_a in fact we have

$$\sum_{a \in A} \rho_a = \sum_{a \in A} \sum_{\alpha(i)=a} \varphi_i = \sum_i \varphi_i = 1$$

and using the observation made at the beginning of the proof

$$\text{supp}\rho_a \subset \bigcup_{\alpha(i)=a} \text{supp}\varphi_i \subset C_a$$

□

To extend this result to every manifold we need some results form topology replacing the compactness condition, namely a locally finite refinement (**).

Lemma 3.8. *Every top mfd admits a countable locally finite cover by precompact (meaning the closure is compact) open sets.*

We now use this result to prove the following

Proposition 3.9. *Consider a smooth manifold M. Every open cover $\{C_i\}$ for M admits refinement, that is another open cover $\{W_a\}$ such that for each W_a I can find a C_i so that $W_a \subset C_i$, satisfying:*

- a *$\{W_a\}$ is countable and locally finite*
- b *I can always find a diffeomorphism $\psi_a : W_a \to B_3(\mathbf{0})$*
- c *$U_a := \{\psi_a^{-1}(B_1(\mathbf{0})\}$ still cover M*

that is $\{W_a\}$ is a regular refinement of $\{C_i\}$.

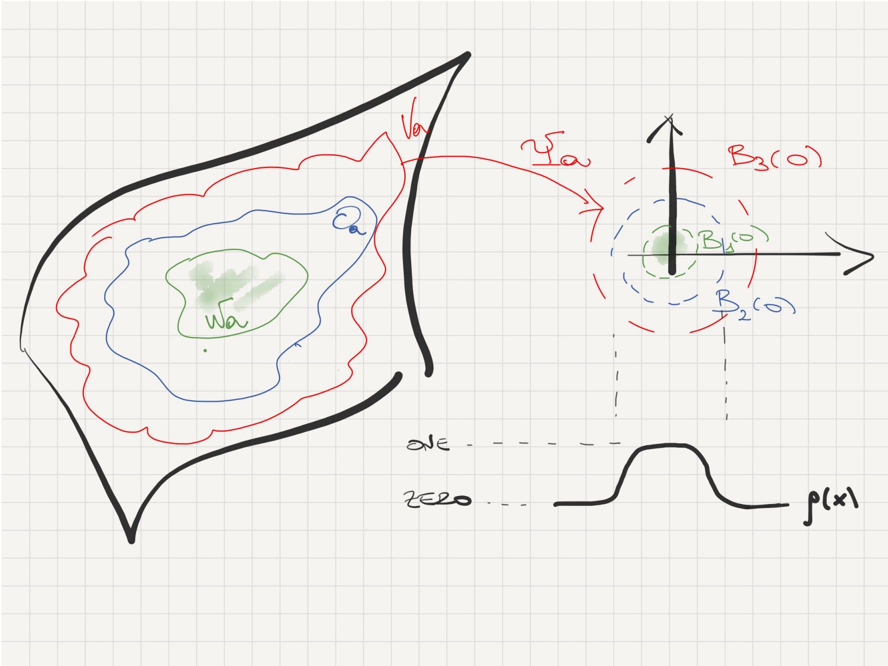
comment *The choice $B_3(\mathbf{0})$ and $B_1(\mathbf{0})$ used in the previous definition may look a bit strange; We will need it in the next theorem, and it is a choice made in order to avoid to write balls with increasing radius $r < r' < r''$*

Proof. See for example Lee book "introduction to smooth manifolds". The compact manifold case is some how simpler but less instructive and can be found in W. Tu book "introduction to manifolds".

Let $\mathfrak{C} = \{C_i\}$ an open cover for M. Consider $\{V_j\}$ a countable locally finite cover for M by precompact open sets that we know exists by the previous Lemma. Then for each $p \in M$ construct (W_p, ψ_p) a coordinate chart centred at p so that

- $\psi_p(W_p) = B_3(\mathbf{0})$
- for every p we can find a C_i so that $W_p \subset C_i$
- When $p \in V_j$ then $W_p \subset V_j$ (**comment** this is possible because the local finiteness of $\{V_j\}$ that is each $p \in M$ has a neighbourhood intersecting finitely many of the V_j).

Define now $U_p = \psi_p^{-1}(B_1(\mathbf{0}))$, and observe that for every $p \in V_j$ the set $\{U_p\}$ is an open cover for $\bar{V}_j$ and by paracompactness of $\bar{V}_j$ we have taht I can a finite subset of those covering it that we call $\{U_1^{(j)}, .., U_{n(j)}^{(j)}\}$ with corresponding $\{W_1^{(j)}, .., W_{n(j)}^{(j)}\}$. The set $(W_i^{(j)}, \psi_i^{(j)})$ refines $\mathcal{C}$ and satisfies b) and c) by construction. We need to prove a) .It is obviously countable, is it also locally finite??? This is a direct consequence of the



fact that $\{V_j\}$ is a countable locally finite cover of M. We refer to the afro mentioned books for more details. □

Theorem 3.10. *Given a smooth mfd M and an open cover $\mathfrak{C} = \{C_i\}$ we can always find a partition of unity subordinate to $\mathfrak{C}$*

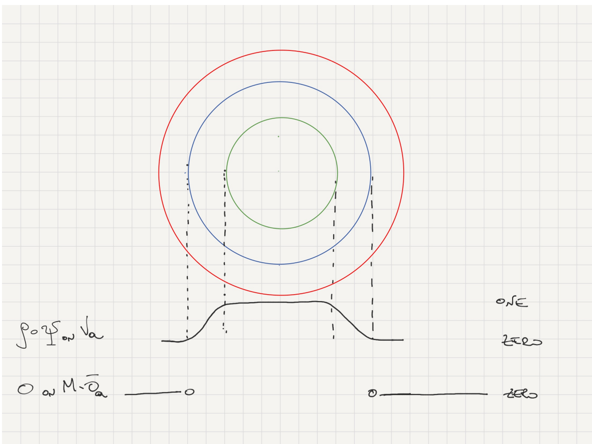
Proof. Consider a regular refinement $\{W_a\}|_{a\in A}$ of $\{C_i\}$ where A is just a set (we need to specify it now). Define then

$$U_a = \psi_a^{-1}(B_1(\mathbf{0}))\,, \quad O_a := \psi_a^{-1}(B_2(\mathbf{0}))$$

Note that by hp $\{W_a\}$ still covers M. We then use the bumb function ρ constructed previously to construct:

$$f_a := \begin{cases} \rho \circ \psi_a & \text{on} \quad W_a \\ 0 & \text{on} \quad M \setminus \bar{O}_a \end{cases}$$

Note that supp$f_a \subset W_a$. Define then



$$g_a(p) = \frac{f_a(p)}{\sum_a f_a(p)}$$

It is here that the regular refinement property plays a crucial role: the denominator contains in fact finitely many non vanishing terms being $\{W_a\}$ locally finite. Observe then that $f_a = 1$ on U_a by construction, and since $\{U_a\}$ cover M by the regularity of the refinement every point is contained in some U_a and we can conclude that the denominator never vanishes and $\sum_a g_a = 1$ and $0 \leq g_a(p) \leq 1$ for all p in M. Now we need to reindex to go back to the given cover $\{C_i\}$. Given C_i we can find a subset $A(i)$ of A so that $U_a \in c_i$ for all $a \in A(i)$ and define

$$\phi_i = \sum_a g_a \,, \quad a \in A(i)$$

Now one can check this function satisfies the desired request. In particular note that $\{\text{supp}(\phi_i)\}$ is locally finite because $\{U_a\}$ is locally finite and $\text{supp}(f_a) \subset U_a$ thus $\text{supp}(\phi_i) \subset C_i$ □

4. Tangent and cotangent bundle

We first comment on tangent vectors on $\mathbb{R}^n$ viewed as derivations and then we generalize. Let's take $\mathbb{R}^2$ equipped with the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ and the origin. Let's consider then the space of vectors applied at a general point $p \in \mathbb{R}^2$. We call this vector space $\mathbb{R}^2_p$ and its general element $\mathbf{v}_p$, that is the vector $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2$ applied at p. $\mathbb{R}^2_p$ is naturally equipped with the standard basis $\{\mathbf{e}_{1p}, \mathbf{e}_{2p}\}$ so that one trivially obtains $\mathbf{v}_p = v_1\mathbf{e}_{1p} + v_2\mathbf{e}_{2p}$. Vector applied to a point are useful to take directional derivative. In particular, given a smooth function f a point p and a vector $\mathbf{v}$ we can define

$$\tilde{\mathbf{v}}_p \cdot f := \frac{d}{dt} f(p + t\mathbf{v}_p)|_{t=0} = \left(v_i \frac{\partial}{\partial x^i} f \right)(p)$$

It is sometimes useful to use the notation $\partial_i|_p f$ to indicate $\left(\frac{\partial}{\partial x^i} f\right)(p)$

Definition 4.1. A lin map $X_p : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ (that we will denote by $X_p \cdot f$ or simply $X_p f$) and satisfying the Leibniz rule

$$X_p(fg) = f(p) X_p \cdot g + g(p) X_p \cdot f$$

is called a derivation at p.

properties:

(1) f constant $\Rightarrow X_p \cdot f = 0$
That's easy to prove, consider the function $f = 1$ then by Leibniz one has $X_p \cdot f = X_p \cdot (f^2) = 2f(p) X_p \cdot f = 2 X_p \cdot f$ from which $X_p \cdot f = 0$

(2) $f(p) = g(p) = 0 \Rightarrow X_p \cdot (fg) = 0$

The space of all derivation at a point p in $\mathbb{R}^n$ is naturally a vector space we will denote by $T_p\mathbb{R}^n$; directional derivatives at a point $\tilde{\mathbf{v}}_p$ are derivations. Natural question: are every derivations of this form??

Proposition 4.2. *$\mathbb{R}^n_p$ and $T_p\mathbb{R}^n$ are isomorphic vector spaces*

Proof. Consider The map $\mathbf{v}_p \to \tilde{\mathbf{v}}_p := v^i \partial_i|_p$. Consider the function $f = x^j$, then we have

$$\tilde{\mathbf{v}}_p x^j = v^i \partial_i|_p x^j = v^j$$

Suppose now $\tilde{\mathbf{v}}_p$ is the zero derivation, the previous relation teaches us that v^j must vanishes; repeating the same argument for every fixed j we end up with $\mathbf{v} = \mathbf{0}$, thus the kernel of this map is only the null vector. Let's check surjectivity. Consider a general $X_p \in T_p\mathbb{R}^n$ and let's use Taylor expansion around the point p with coordinate $(x_0^1, .., x_0^n)$ to write a general smooth function f as follows:

$$f(\mathbf{x}) = f(p) = \partial_i f(p)(x^i - x_0^i) + g_i(\mathbf{x})(x - x_0^i)$$

for some smooth functions g_i vanishing in p. We now use the Leibniz rule and we get

$$X_p \cdot f = \underbrace{X_p \cdot f(p)}_{0 \text{ by prop. } 1} + X_p \cdot (\partial_i f(p)(x^i - x_0^i)) + \underbrace{X_p \cdot g_i(\mathbf{x})(x^i - x_0^i)}_{0 \text{ bc of property } 2} = (\partial_i f)(p) X_p (x^i - x_0^i)|_{x^i = x_0^i}$$

Defining now $X_p \cdot x^i := v^i$ we obtain that every X_p comes from a vector $\mathbf{v} = v^i \mathbf{e_i}$ □

Corollary 4.3. *$\{\partial_i|_p\}$ is a basis for $T_p\mathbb{R}^n$*

A (smooth) vector field X on an open subset U of $\mathbb{R}^n$ is a function that assigns to each point p in U a tangent vector X_p in $T_p\mathbb{R}^n$. Every element of this space can be written as:

$$X = a^i(\mathbf{x})\partial_i$$

with a^i smooth functions. In particular for every $p \in U$ one has

$$(X \cdot f)(p) = X_p \cdot f = a^i(p)\partial_i|_p f$$

We aim now to generalize to a general smooth manifold and many results and definition discussed previously generalize naturally to this setting.

Definition 4.4. A lin map $X_p : C^\infty(M) \to \mathbb{R}$ (that we will denote by $X_p \cdot f$ or simply $X_p f$) and satisfying the Leibniz rule

$$X_p \cdot (fg) = f(p) X_p \cdot g + g(p) X_p \cdot f$$

is called a derivation at p.

Properties 1) and 2) discussed previously still hold. There is an alternative definition some how more geometrical that uses the concept of tangent vectors to curves. In order to carefully introduyce this point of view we need more mathg stuctures.

Suppose now we have a map $F : M \to N$, then, for every $p \in M$, it naturally induces a linear map $F_* : T_pM \to T_{F(p)}N$ by

$$(F_* X_p)_{F(p)} \cdot f := X_p \cdot (F \circ f)$$

is called push forward. Sometimes it is useful to specify the initial point and we are forced to modify the notetion and use dF_p or $F_*|_p$ instead of just F_*. We will sometimes write just $F_* X_p$ instead of $(F_* X_p)_{F(p)}$ in order to simplify the notation.

Push forward properties:

- $(G \circ F)_* = G_* \circ F_*$
- F a diffeomorphism $\Rightarrow F_*$ an isomorphism

It is interesting to observe how the definition of tangent space it is actually a local construction even if we have used smooth functions on M to define it and not on an neighborhood U of the point p. Let's justify it by using the following Lemma:

Lemma 4.5. *If $f, g \in C^\infty(M)$ agree on some neighborhood U of $p \in M$ then*

$$X_p \cdot f = X_p \cdot g$$

Proof. Take the closed subset $A := M \setminus U$ and consider the bumb function β for A supported in $M \setminus \{p\}$ and define $h = f - g$. Note now that in U we have that $h = 0$ and and A we have $\beta = 1$, combining this two infos we have that $h(q) = h(q)\beta(q)$ for every $q \in M$ thus:

$$X_p \cdot (h) = X_p \cdot (h\beta) \overset{Leibniz}{=} h(p)X_p \cdot \beta + \beta(p)X_p \cdot h$$

that vanishes because $h(p) = \beta(p) = 0$. Thus by linearity on every point in U we have $X_p \cdot f = X_p \cdot g$ □

Fact: The tangent space at a point p of a finite dimensional vector space V, that is T_pV, is isomorphic to V, namely

$$T_pV \sim V\,, \quad \forall p \in V$$

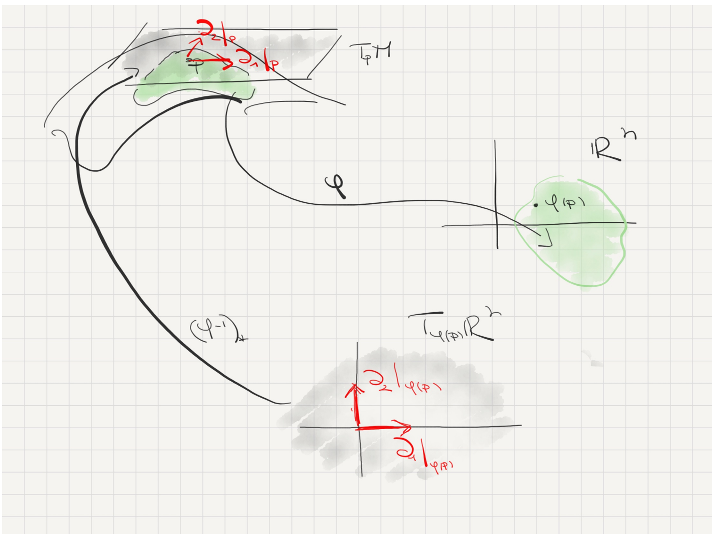
We were able so far to write functions representation in a chart, we will do the same for derivations. Consider the chart (U, φ) for M. We know we have a canonical basis for $T_{\hat{p}}\mathbb{R}^n$ given by $\partial_i|_{\hat{p}}$ and that, being φ a diffeomorphism we know by the second property of the push forward that $T_{\hat{p}}\mathbb{R}^n \sim T_pM$ thus the set $\{\partial_i|_p\}$ defined by

$$\partial_i|_p := (\varphi^{-1})_*\partial_i|_{\hat{p}}$$

is a basis for T_pM. What's the meaning of this object?? Consider a function$f : U \to \mathbb{R}$ then

$$\partial_i|_p f = (\varphi^{-1})_*\partial_i|_{\hat{p}} f = \partial_i|_{\hat{p}}(f \circ \varphi^{-1}) = \partial_i|_{\hat{p}}\hat{f} = \frac{\partial \hat{f}}{\partial x^i}(\hat{p})$$

Let 's see now how to pushforward this basis. Consider two charts (U, φ) and (V, ψ) for



M and N and denote the coordinates in the domain by $\hat{p} = \varphi(p) = (x^1, ..., x^n) = x^i$ and

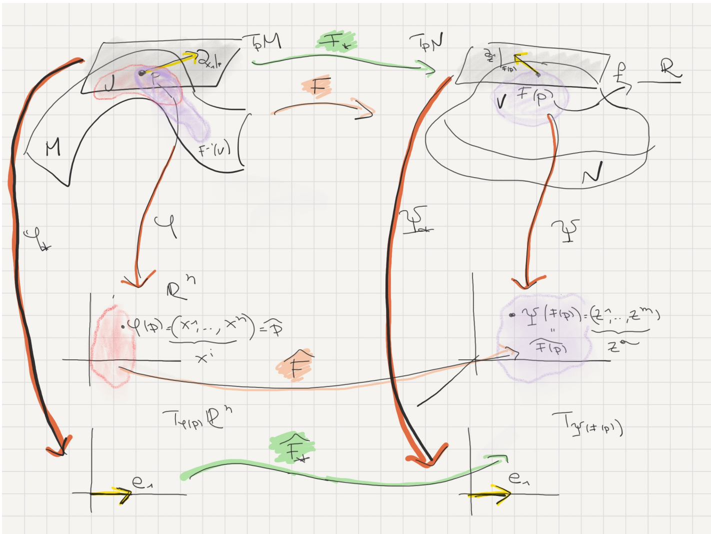
$\hat{q} = \psi(q) = (z^1, ..., z^m) = z^a$, take then consider $f \in C^\infty(N)$:

$$
\begin{aligned}
(F_*(\partial_{x^i}|_p))f &= (\partial_{x^i}|_p))(F \circ f) \\
&= (\partial_{x^i}|_{\hat{p}}))(f \circ F \circ \varphi^{-1}) \\
&= (\partial_{x^i}|_{\hat{p}}))(\underbrace{f \circ \psi^{-1}}_{\hat{f}} \circ \underbrace{\psi \circ F \circ \varphi^{-1}}_{\hat{F}}) \\
\text{chain rule} \quad &= \frac{\partial \hat{F}^a}{\partial x^i}(\hat{p})(\partial_{z^a}\hat{f})(\widehat{F(p)}) \\
&= \frac{\partial \hat{F}^a}{\partial x^i}(\hat{p})\, \partial_{z^a}|_{F(p)} f
\end{aligned}
$$

from which we get

$$
(F_*(\partial_{x^i}|_p)) = \frac{\partial \hat{F}^a}{\partial x^i}(\hat{p})\, \partial_{z^a}|_{F(p)}
$$

We have seen that in a given coordinate chart (U, φ) with coordinates function of φ



given by $(x^1, .., x^n)$, we have a natural basis for T_pM thus every tangent vector at p can be written as

$$
X_p = a^i \partial_i|_p = a^i \frac{\partial}{\partial x^i}\bigg|_p
$$

What if we change chart (V, ψ) with coordinate function $(\tilde{x}^1, .., \tilde{x}^n)$? (note we take $p \in V \cap U$). In this case we can use the basis $\{\tilde{\partial}_i|_p\}$ with $\tilde{\partial}_i|_p := \frac{\partial}{\partial \tilde{x}^i}|_p$ and we have

$$
X_p = \tilde{a}^i \tilde{\partial}_i|_p
$$

How the tilde and non tilde are related?? First of all let's write the transition map:

$$
\psi \circ \varphi^{-1}(\mathbf{x}) = (\tilde{x}^1(\mathbf{x}), .., \tilde{x}^n(\mathbf{x})) = \tilde{\mathbf{x}}(\mathbf{x})
$$

Let's compute it by acting on a general smooth function f:

$$\begin{aligned}
\partial_i|_p\, f &\overset{def}{=} ((\varphi^{-1})_*\, \partial_i|_{\varphi(p)})\, f \\
&= (\partial_i|_{\varphi(p)})\, \underbrace{f \circ \varphi^{-1}}_{\hat{f}(\mathbf{x})} \\
&= (\partial_i|_{\varphi(p)})\, \underbrace{f \circ \psi^{-1}}_{\hat{f}(\tilde{\mathbf{x}})} \circ \underbrace{\psi \circ \varphi^{-1}}_{\tilde{\mathbf{x}}(\mathbf{x})} \\
\text{chain rule} &= \left(\frac{\partial \tilde{x}^j}{\partial x_i}\bigg|_{\varphi(p)} \tilde{\partial}_j|_{\psi(p)} \right) \hat{f}(\tilde{\mathbf{x}}) \\
&\overset{def}{=} \left(\frac{\partial \tilde{x}^j}{\partial x_i}\bigg|_{\varphi(p)} \tilde{\partial}_j|_p \right) f
\end{aligned}$$

from which we get

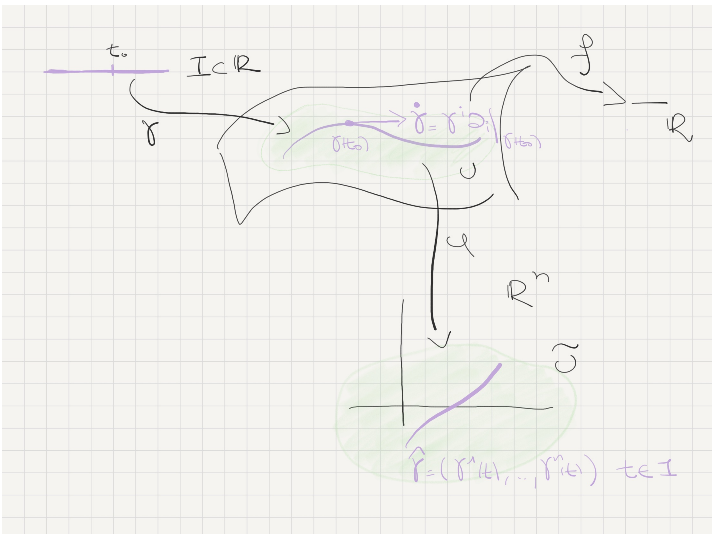
$$\partial_i|_p = \frac{\partial \tilde{x}^j}{\partial x_i}\bigg|_{\varphi(p)} \tilde{\partial}_j|_p$$

and thus

$$a^i = \frac{\partial x^i}{\partial \tilde{x}^j}\bigg|_{\varphi(p)} \tilde{a}^j$$

There is a more geometrical interpretation of tangent vectors we are now able to discuss properly

Definition 4.6. A smooth curve is a smooth map $\gamma : J \subset \mathbb{R} \to M$; we will often denote it by $\gamma(t)$



Take $t_0 \in J$, define $p_0 = \gamma(t_0)$ and f smooth function on M; consider then the function restricted to the curve, that is $f(\gamma(t))$, let's see how it changes

$$\left(\frac{d}{dt}(f \circ \gamma)\right)(t_0) = (\gamma_* \partial_t|_{t_0}) f$$

In a given coordinate chart containing $\gamma(t_0)$ we know how to handle this object and we get

$$(\gamma_* \partial_t|_{t_0})_{\gamma(t_0)} f = \frac{d}{dt}\Big|_{t_0} f \circ \gamma = \frac{d}{dt}\Big|_{t_0} (\underbrace{f \circ \varphi^{-1}}_{\hat{f}} \circ \underbrace{\varphi \circ \gamma}_{\hat{\gamma}}) = \frac{d}{dt}\Big|_{t_0} \hat{f}(\gamma^1(t), .., \gamma^n(t))$$

where $\hat{\gamma} = (\gamma^1(t), .., \gamma^n(t))$ simply means we are taking the coordinate representation of every point along the curve. In conclusion we obtain:

$$\dot{\gamma}^i(t_0)\partial_i|_{\hat{\gamma}(t_0)}\hat{f} = \dot{\gamma}^i(t_0)\partial_i|_{\gamma(t_0)} f$$

with $\dot{\gamma}^i := \partial_t \gamma^i$. We will sometimes denote this vector by $\dot{\gamma}_{\gamma(t_0)}$ In conclusion given a curve γ we can produce a tangent vector at $p_0 = \gamma(t_0)$ that is $\dot{\gamma}^i(t_0)\partial_i|_{p_0}$.
Natural question: every tangent vector arise from a curve?

Lemma 4.7. *Let $p \in M$ then every $X_p \in T_pM$ is the tangent vector to some curve*

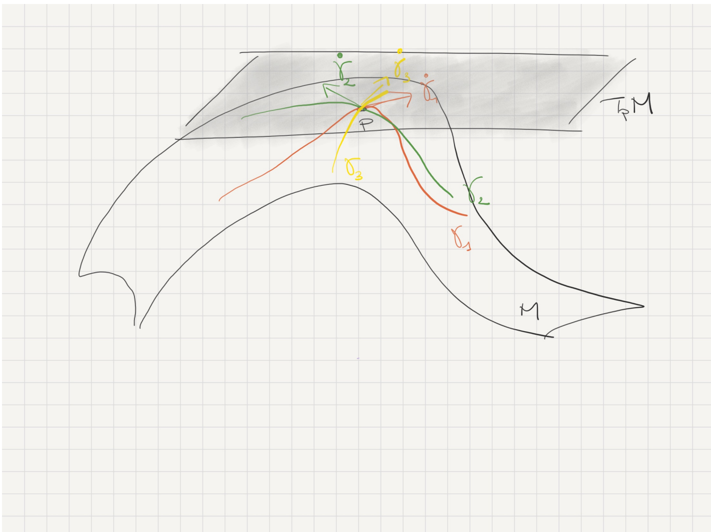
Proof. Consider a coordinate chart (U, φ) centered at p (that is $\varphi(p) = \mathbf{0}$) the vector $X_p = a^i\partial_i|_p$. We want to construct a curve $\gamma : (-\epsilon, \epsilon) \to U$ such that $\dot{\gamma}^i(0) = a^i$ and $\gamma(0) = p$. To this aim we construct $\hat{\gamma} = (ta^1, .., ta^n)$ from which we have

$$\gamma(t = 0) = \varphi^{-1}(\hat{\gamma}(t = 0)) = \varphi^{-1}(\mathbf{0}) = p$$

and

$$\dot{\gamma}_{\gamma(0)} = \dot{\gamma}^i(0)\partial_i|_{\gamma(0)} = a^i\partial_i|_{\gamma(0)=p}$$

□



5. Vector bundles

We want now to extend the notion of vector field on $\mathbb{R}^n$ to a general smooth manifold M, namely something that evaluated at each point p produces an element of T_pM "smoothly". This goal can be achieved using the general notion of vector bundle that we introduce int he following:

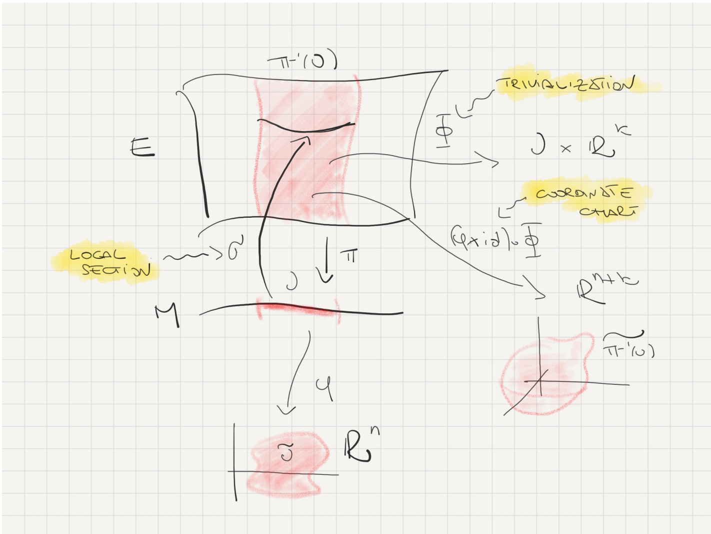
Definition 5.1. Consider M and E smooth manifolds a a smooth surjective map $\pi : E \to M$. If

- $E_p \stackrel{def}{=} \pi^{-1}(p)$ named the fiber over p, is a real vector space of dimension k
- for every $p \in M$ we can find a neighborhood of p and a diffeomorphism

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$$

  s.t. $\Phi|_{\pi^{-1}(p)} : E_p \to \{p\} \times \mathbb{R}^k \sim \mathbb{R}^k$ is a linear isomorphims and $pr_1 \circ \Phi = \pi$. The diffeo ϕ satisfying this requirements is called local trivialization

we will call the set of data $(E, M, \pi, \{(U, \Phi)\})$ a (smooth) vector bundle of rank k (we will just denote it by $\pi : E \to M$ or jut E sometimes in the future).



What happen when we change trivialization?

Lemma 5.2. *Given two trivialization (U, Φ) and (V, Ψ) of a vector bundle of rank k with $U \cap V \neq \emptyset$, then we can find a smooth map $\mathbf{g} : U \cap V \to GL_k(\mathbb{R})$ s.t. for every $p \in U \cap V$ we have*

$$\Phi \circ \Psi^{-1}(p, \mathbf{v}) = (p, \mathbf{g}(p) \cdot \mathbf{v})$$

where we denoted by $\mathbf{g}(p) \cdot \mathbf{v}$ the action of an element of $GL_k(\mathbb{R})$ on an arbitrary element $\mathbf{v} \in \mathbb{R}^k$

Proof. We know that $pr_1 \circ \Phi = \pi = pr_1 \circ \Psi$ thus the following diagram commutes

$$\begin{array}{ccccc} (U \cap V) \times \mathbb{R}^k & \xrightarrow{\Psi^{-1}} & \pi^{-1}(U \cap V) & \xrightarrow{\Phi} & (U \cap V) \times \mathbb{R}^k \\ & \searrow^{pr_1} & \downarrow \pi & \swarrow_{pr_1} & \\ & & U \cap V & & \end{array}$$

then we can conclude that

$$pr_1 \circ \Phi \circ \Psi^{-1} = pr_1 \quad \longrightarrow \quad \Phi \circ \Psi^{-1}(p, \mathbf{v} = (p, \nu(p, \mathbf{v}))$$

for some $\nu : (U \cap V) \times \mathbb{R}^k \to \mathbb{R}^k$ that is smooth by construction. Fixed $p \in M$ we know that the trivialization map is a lin. isomorphism thus is $\nu(p, \mathbf{v}) = \mathbf{g}(p) \cdot \mathbf{v}$ (thus

$\mathbf{g}(p) \in GL_k(\mathbb{R}))$. Smoothness of ν easily implies smoothness of $\mathbf{g}$. Choose a basis $\{\mathbf{b}_a\}$ for E_p with dual basis $\{\boldsymbol{\beta}^a\}$ then we can write explicitly

$$\mathbf{g}(p) \cdot \mathbf{v} = g^a{}_b(p) v^b \mathbf{b}_a$$

from which one gets that $g^a{}_b(p) = \boldsymbol{\beta}^a(\nu(p, \mathbf{b_b}))$ that is $g^a{}_b(p)$ are smooth functions being the composition of smooth functions. □

Definition 5.3. A local section of a vector bundle is a smooth map $\sigma : U \subset M \to E$ s.t. $\pi \circ \sigma = id_U$. In the case $U = M$ we will call it global section and we will denote them by $\Gamma(E)$

A set of local (global) sections $\{\sigma_1, .., \sigma_k\}$ s.t. for each $p \in U$ (or every point in M) we have that

$$\{\sigma_1(p), .., \sigma_k(p)\}$$

is a basis for E_p, is called local (global) frames.

Notation: It is common to identify a section of a rank k vector bundle at a point, in a given trivialization, with an element of $\mathbb{R}^k$, that is

$$\sigma^\Phi(p) := \Phi \circ \sigma(p) = (p, \mathbf{v}) \sim \mathbf{v} \in \mathbb{R}^k$$

Observation: being the trivialization a pointwise isomorphism we can conclude that local frames and trivialization are in 1-1 correspondence (see the exercise sheet).

Definition 5.4. The disjoint union

$$TM := \sqcup_p T_p M$$

is named tangent bundle

We need to prove it deserve the name bundle by showing that:

(1) TM is a smooth manifold
(2) $\pi : TM \to M$ is a smooth surjective map
(3) it could be equipped with a canonical set of local trivializations.

prove 1: Given a coordinate chart (U, φ) for M we can construct

$$\begin{array}{rcrcl} \Phi & : & \pi^{-1}(U) \subset TM & \to & (\tilde{U}) \times \mathbb{R}^n \subset \mathbb{R}^{2n} \\ & & (p, a^i \partial_i|_p) & \mapsto & (\underbrace{x^1, .., x^n}_{\mathbf{x}=\hat{p}}, a^1, .., a^n) \end{array}$$

it's a bijection (when restricted to its image) with inverse map

$$\begin{array}{rcrcl} \Phi^{-1} & : & \tilde{U} \times \mathbb{R}^n & \to & \pi^{-1}(U) \subset TM \\ & & (\underbrace{x^1, .., x^n}_{\mathbf{x}=\hat{p}}, a^1, .., a^n) & \mapsto & (\varphi^{-1}(\mathbf{x}), a^i \partial_i|_{\varphi^{-1}(\mathbf{x})}) \end{array}$$

We want to use now Φ to transfer the topology of $\varphi(U) \times \mathbb{R}^n \subset \mathbb{R}^{2n}$ to TU .

We declare a set A in $TU := \pi^{-1}(U)$ open if its image by Φ is open in $\varphi(U) \times \mathbb{R}^n \subset \mathbb{R}^{2n}$. Observe that if V is a (open) subset of U then the subset topology of TV coincide with the one given by the trivialization map restricted to TV. Consider now an Atlas with coordinate open set $\gamma := \{U_i\}$ and let's construct

$$\mathcal{C} := \{A \text{ s.t. } A \text{ is open in } TU_i \text{ with } U_i \in \gamma\}$$

that is the collection of all open subsets of all TU_i.

Lemma 5.5. *Take U_1 and U_2 coordinate open sets in M and A_1 and A_2 opens inside TU_1 and TU_2, then $A_{12} := A_1 \cap A_2$ is open in $T_{12} := T(U_{12})$ where $U_{12} = U_1 \cap U_2$.*

Proof. Note then that $A_1 \cap T(U_{12})$ and $A_2 \cap T(U_{12})$ are opens in $T(U_{12})$ by definition of subset topology; note then that $A_{12} \subset TU_1 \cap TU_2 = T_{12}$ hence

$$A_{12} = A_1 \cap A_2 = A_1 \cap A_2 \cap T_{12} = (A_1 \cap T(U_{12})) \cap (A_2 \cap T(U_{12}))$$

thus is open because intersection of opens. □

By subspace topology and $T(U_{12})$ is open in $T(U_1)$ we have that A_{12} for example is open in TU_1 and thus belong to the given collection. It is easy to see that the union of all A in $\mathcal{C}$ gives TM. This observation and the previous lemma shows that $\mathcal{C}$ satisfies the condition of Lemma (1.3) that we report here for convenience:

A collection $\beta = \{u_i\}$ of subset of S is a basis for some topology of S if and only if

- S is the union of all u_i
- given u_i and u_j and $p \in u_i \cap u_j$ there is $u_k \in \beta$ s.t. $p \in u_k \subset u_i \cap u_j$

It wold be in fact enough to take for every p in $A_1 \cap A_2$ the open A_{12} that is in $\mathcal{C}$ as shown before. Thus we can conclude that on TM we can induce the topology given by this basis

Lemma 5.6. *(*) A manifold M has a countable basis consisting of coordinate open sets.*

Proof. Let $\{(U_a, \varphi_a)\}$ be a maximal atlas on M and $\gamma = \{u_i\}$ a countable basis for M. For each $p \in M$ and U_a containing p we can find an elment of γ we name $u_{p,a}$ s.t. $p \in u_{p,a} \subset U_a$ (essentailly by the definition of basis for the topology). The collection $\{u_{p,a}\}$ without duplicate elements is a subcollecition of β and thus second countable. Is it a basis? For any open set U and $p \in U$ we can find U_a s.t.

$$p \in U_a \subset U$$

thus

$$u_{p,a} \subset U$$

which shows that $\{u_{p,a}\}$ is a basis for a topology on M. □

Proposition 5.7. *(*) TM is second contable*

Proof. Consider a countable basis of coordinate opens sets $\{U_i\}$. Being TU_i homeomorphic to an open subset of $\mathbb{R}^{2n}$ is second countable. For each TU_i choose a countable basis, and thus also TM will be second countable being the basis for M chosen countable. □

It is then easy to prove that TM is Hausdorff. If (p, X_p) and (q, X_q) live on two different fibers then we can find two neighborhoods for p and q we call U_1 and U_2 separating the points so that TU_1 and TU_2 separates (p, X_p) and (q, X_q). If we consider points on the same fiber we can easily use the isomorphism with $\mathbb{R}^n$

Moreover TM is also locally Euclidean by construction and thus a topological manifold.

We know what happens when we change coordinate chart from (U_i, φ_i) with $\varphi_i(p) := \mathbf{x} = (x_1, .., x_n)$ to (U_j, φ_j) with $\varphi_j(p) := \tilde{\mathbf{x}} = (\tilde{x}_1, .., \tilde{x}_n)$ on M and T_pM, putting these stuff together we obtain

$$\Phi_j \circ \Phi_i^{-1}(x^1, .., x^n, a^1, .., a^n) = (\tilde{x}^1(\mathbf{x}), .., \frac{\partial \tilde{x}^1(\mathbf{x})}{\partial x^i} a^i, ...)$$

that is clearly smooth. We thus can construct in this way a smooth atlas $\{TU_i, \Phi_i\}$ and thusa smotth structure.

prove 2: By construction

prove 3: It is easy to show that $\tilde{\Phi} = (\varphi^{-1} \times id_{\mathbb{R}^n}) \circ \Phi : \pi^{-1}(U) \subset TM \to U \times \mathbb{R}^n$ is a trivialization map

Definition 5.8. A (smooth) vector field is a section of TM. Sometimes one might be interested in maps from M to TM not necessary smooth or continuous. Here we will consider only smooth maps thus (smooth) sections of TM unless specified. The standard notation for $\Gamma(TM)$ is $\mathfrak{X}(M)$

Lemma 5.9. *In every coordinate chart, the component functions of $X \in \mathfrak{X}(M)$ are smooth functions. Moreover a vector field, for any $f \in C^\infty(M)$, naturally define the function $X \cdot f : M \to \mathbb{R}$ defined by $(X \cdot f)(p) = X_p \cdot f$; this one is smooth.*

Proof. Given the chart (U, φ) with coordinate function $\mathbf{x} = (x^1, .., x^n)$ for M we construct the coordinate rep of $X : M \to TM$

$$\hat{X}(\mathbf{x}) = (x^1, , .x^n, \hat{a}^1(\mathbf{x}), .., \hat{a}^n(\mathbf{x}))$$

thus the component functions $\hat{a}^i$ on U must be smooth functions, thus in the given coordinate chart the coordinate representation $\widehat{X \cdot f}$

$$\widehat{X \cdot f}(\mathbf{x}) = \hat{a}^i(\mathbf{x})(\partial_i \hat{f})(\mathbf{x})$$

that is obviously smooth being $\hat{a}^i$ and $\partial_i f$ smooth functions. For this reason we think at a vector field in a given coordinate chart as something of the form $a^i \partial_i$ with a^i smooth functions on M □

6. The cotangent bundle

Take now the dual space of T_pM; it is called the cotangent space and denoted by T_p^*M. In a given coordinate open set$U$, with coordinate functions $(x^1, .., x^n)$ we have the coordinate basis for T_pM given by $\{ \frac{\partial}{\partial x^i}\big|_p \}$, and we denote the dual one by $\{ dx^i\big|_p \}$. Thus every element of T_pM can be written as $\omega_p = \omega_i dx^i\big|_p$. In particular we note that, being $dx^i\big|_p (\partial_j|_p) = \delta^i_j$ one has that

$$\omega_i = \omega_p(\partial_i|_p)$$

What happens when we change the coordinate chart? denoting as usual the new coordinate function by $(\tilde{x}^1, .., \tilde{x}^n)$ we know that

$$\partial_i|_p = \left.\frac{\partial \tilde{x}^j}{\partial x^i}\right|_{\varphi(p)} \tilde{\partial}_j|_p$$

thus

$$dx^i|_p = \left.\frac{\partial x^i}{\partial \tilde{x}^j}\right|_{\varphi(p)} d\tilde{x}^j|_p$$

Let us resume for convinience what we obtain so far:

$$
\begin{aligned}
(x^1, .., x^n) &\to (\tilde{x}^1, .., \tilde{x}^n) \\
\partial_i|_p &\to \tilde{\partial}_i|_p = \frac{\partial x^j}{\partial \tilde{x}^i}\Big|_{\varphi(p)} \partial_j|_p \\
a^i &\to \tilde{a}^i = \frac{\partial \tilde{x}^i}{\partial x^j}\Big|_{\psi(p)} a^j \\
dx^i|_p &\to d\tilde{x}^i|_p = \frac{\partial \tilde{x}^i}{\partial x^j}\Big|_{\psi(p)} dx^j|_p \\
\omega_i &\to \tilde{\omega}_i = \frac{\partial x^j}{\partial \tilde{x}^i}\Big|_{\varphi(p)} \omega_j
\end{aligned}
$$

Definition 6.1. The disjoint union

$$T^*M := \sqcup_p T_p^*M$$

is named cotangent bundle

Proposition 6.2. *T^*M has a natural stucture of a vector bundle of rank n over M*

Proof. The proof mimic the one discussed previously for TM □

Definition 6.3. Smooth sections of T^*M, namely $\Gamma(T^*M)$, are named covector fields on M. Covector fields are sometimes called differential one forms and the set of covector fields is also denoted by $\Omega^1(M)$. Given a covector field ω and a smooth function f then $f\omega$ is a covector field naturally defined by $(f\omega)_p = f(p)\omega_p$

Lemma 6.4. *Let $\omega \in \Omega^1(M)$, then*

- *In any coordinate chart for M and in the coordinate basis for T_p^*M we will write $\omega := \omega_i dx^i$. The components ω_i of the coordinate representation for ω are smooth functions.*
- *Given a smooth vector field X one has that $\omega(X)(p) = \omega_p(X_p)$ is smooth.* [2]

Given a smooth function we can naturally produce a covector field *df* called the differential of f.The symbol d will be also discussed in the following it willplay a crucial role; at thsi stage let s just view *df* as a unique symbol ; *df* is a covector field defined such that

$$df_p(X_p) = X_p \cdot f$$

In a given coordinate chart we have

$$df_p(\partial_i|_p) = (\partial_i f)(p)$$

so locally we may write $df = \partial_i f dx^i$. Note that by construction *df* is a smooth map from M to T^*M. The "dual" manouvre with respect to the pushforward is named pullback: give the smooth map $F : M \to N$ we construct

$$F^* : T^*_{F(p)}N \to T_p^*M$$

defined by

$$(F^*\omega_{F(p)})(X_p) = \omega_{F(p)}(F_*X_p)$$

[2]Remember we use the notation: $\omega(p) = \omega_p$ and $X(p) = X_p$

Observation If $F : M \to N$ is as usual a smooth map and $\omega \in \Omega^1(N)$ then $F^*\omega \in \Omega^1(M)$. It is easy to check in a coordinate system. Note that with respect to the vector field case we don't have an ambiguity (that can be caused for example by an non injective map) and everything is well defined since since we are pulling back objects. We will sometimes use the following notations to specify the same thing

$$(F^*(\omega_{F(p)}))_p = (F^*\omega)_p = F^*(\omega_{F(p)})$$

Lemma 6.5. *Let $F : M \to N$ a smooth map, $f \in C^\infty(N)$ and $\omega \in \Omega^1(N)$*

- $F^*df = d(f \circ F)$
- $F^*(f\omega) = (f \circ F)F^*\omega$

Proof. Let's prove the first one. Take a smooth vector field $X \in \mathfrak{X}(M)$

$$\begin{aligned} (F^*df_{F(p)})_p(X_p) &= df_{F(p)}(F_*X_p)_{F(p)} \\ \text{by definition of differential} &= (F_*X_p)_{F(p)} \cdot f \\ \text{by definition of pushforward} &= X_p \cdot (f \circ F) \\ \text{by definition of differential} &= (d(f \circ F))_p(X_p) \end{aligned}$$

Similarly for the second property one has:

$$\begin{aligned} F^*((f\omega)_{F(p)}) &= F^*(f(F(p)\omega_{F(p)}) \\ &= f(F(p))(F^*\omega_{F(p)}) \end{aligned}$$

□

Observation the previous Lemma can be efficiently used as follows. Consider the identity map from M with a given coordinate chart and M with an other coordinate chart. The pullback can be then used to compute how a given covector changes under a change of coordinate.

7. Tensors

We start defining tensors on a real finite dimensional vector space V and we then generalize the construction to a general manifold.

Definition 7.1. A covariant k tensor is a multilinear map:

$$\tau : \underbrace{V \times .. \times V}_{k \text{ times}} \to \mathbb{R}$$

The set of all covariant k tensor is denoted by T^kV.

A controcovariant p tensor is a multilinear map:

$$Y : \underbrace{V^* \times .. \times V^*}_{p \text{ times}} \to \mathbb{R}$$

The set of all controvariant p tensor is denoted by T_pV.

A mixed tensor of type (k, p) is a multilinear map:

$$F : \underbrace{V \times .. \times V}_{k \text{ times}} \times \underbrace{V^* \times .. \times V^*}_{p \text{ times}} \to \mathbb{R}$$

The set of all mixed tensor is denoted by $T^k_p V$.

Observe that $T^k V$ and $T_p V$ and $T^k_p V$ can be endowed with the structure of a real vector space. We can in principle be more general and for example consider multilinear map from $V \times W \to \mathbb{R}$ with both V and W real finite dimensional vector spaces.

Example 7.2. Consider $\alpha, \beta \in V^*$ and define the map

$$\begin{array}{rcl} \tau_{\alpha,\beta} : V \times V & \to & \mathbb{R} \\ \mathbf{v}_1, \mathbf{v}_2 & \to & \alpha(\mathbf{v}_1)\beta(\mathbf{v}_2) \end{array}$$

This map is obviously multilinear thus $\tau_{\alpha,\beta} \in T^2 V$. Let us remark at this point that, given $a \in \mathbb{R}$

$$a\tau_{\alpha,\beta} = \tau_{a\alpha,\beta} = \tau_{\alpha,a\beta}$$

and

$$\tau_{\alpha+\alpha',\beta} = \tau_{\alpha,\beta} + \tau_{\alpha',\beta}$$

$$\tau_{\alpha,\beta+\beta'} = \tau_{\alpha,\beta} + \tau_{\alpha,\beta'}$$

We will go back to those relations soon.

Take now $\tau \in T^k V$ and $\rho \in T^m V$and define an element of $T^{k+m} V$ denoted by $\tau \otimes \rho$ by

$$(\tau \otimes \rho)(\mathbf{v}_1, .., \mathbf{v}_{k+m}) = \tau(\mathbf{v}_1, .., \mathbf{v}_k)\rho(\mathbf{v}_{k+1}, .., \mathbf{v}_{k+m})$$

It is easy to show that the tensor product is associative.

Particularly interesting covariant k tensor that are symmetric alternating, that is: let $\sigma \in S_k$ a permutation of k object and denote by $sign(\sigma)$ the sign of the permutation.

Definition 7.3. $\rho \in T^k V$ is named symmetric if

$$\rho(\mathbf{v}_1, .., \mathbf{v}_k) = \rho(\sigma(\mathbf{v}_1), .., \sigma(\mathbf{v}_k))$$

and alternating if

$$\rho(\mathbf{v}_1, .., \mathbf{v}_k) = sign(\sigma)\rho(\sigma(\mathbf{v}_1), .., \sigma(\mathbf{v}_k))$$

An example of alternating covariant tensor is the determinant of a matrix, viewed as multilinear map on column vectors. The set of alternating ans symmetric covariant k tensor are denoted by $\Lambda^k V$ and $\Sigma^k V$. Given $\rho \in T^k V$ we can construct two natural map to obtain symmetric or alternating tensor

$$Symm : T^k V \to \Sigma^k V \quad \text{by} \quad Symm(\rho)(\mathbf{v}_1, .., \mathbf{v}_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \rho(\sigma(\mathbf{v}_1), .., \sigma(\mathbf{v}_k))$$

and similarly

$$Alt : T^k V \to \Lambda^k V \quad \text{by} \quad Alt(\rho)(\mathbf{v}_1, .., \mathbf{v}_k) = \frac{1}{k!} sign(\sigma) \sum_{\sigma \in S_k} \rho(\sigma(\mathbf{v}_1), .., \sigma(\mathbf{v}_k))$$

Take now $\tau \in \Lambda^k V$ and $\rho \in \Lambda^m V$; we can construct another element of $\Lambda^{k+m}V$ denoted by $\tau \wedge \rho$ and constructed as follows

$$\tau \wedge \rho = \frac{(k+m)!}{k!m!} Alt(\tau \otimes \rho)$$

Explicitly we have (fix coeff)

$$(\tau \wedge \rho)(\mathbf{v}_1, .., \mathbf{v}_{k+m}) = \frac{(k+m)!}{k!m!} \frac{1}{(k+m)!} \sum_{\sigma \in S_k} \tau(\sigma(\mathbf{v}_1), .., \sigma(\mathbf{v}_k))\rho(\sigma(\mathbf{v}_{k+1}), .., \sigma(\mathbf{v}_{k+m}))$$

Proposition 7.4. *Given ρ and τ as before we have that :*

- *the wedge product is associative*
- $(\tau \wedge \rho) = (-)^{km}(\rho \wedge \tau)$

Proposition 7.5. *Let $\{\mathbf{b}_i\}$ and $\{\boldsymbol{\beta}^i\}$ dual basis for V and V^*. Then set given by all covariant k-tensor or rank k of the forms*

$$\boldsymbol{\beta}^{i_1} \otimes .. \otimes \boldsymbol{\beta}^{i_k}$$

form a basis for $T^k V$

Proof. Take any $\rho \in T^k V$ and define $\rho_{i_1..i_k} := \rho(\mathbf{b}_{i_1}, .., \mathbf{b}_{i_k})$. It is easy now to show that $\rho = \rho_{i_1..i_k}\boldsymbol{\beta}^{i_1} \otimes .. \otimes \boldsymbol{\beta}^{i_k}$.Thus the set $\{\boldsymbol{\beta}^{i_1} \otimes .. \otimes \boldsymbol{\beta}^{i_k}\}$ span $T^k V$. Are they lin indip? Consider the equation

$$\lambda_{i_1..i_k}\boldsymbol{\beta}^{i_1} \otimes .. \otimes \boldsymbol{\beta}^{i_k} = 0$$

and apply it to any sequence $(\mathbf{b}_{j_1}, .., \mathbf{b}_{j_k})$ and we discover that zero can be obtained only taking all the λ to be zero. □

Proposition 7.6. *Given a real finite dimensional vector space V and a basis and dual basis $\{\mathbf{b}_i\}$ and $\{\boldsymbol{\beta}^j\}$, the set $\{\boldsymbol{\beta}_{i_1} \wedge .. \wedge \boldsymbol{\beta}_{i_k}\}$ with $i_1 < i_2 < .. < i_k$ is a basis for $\Lambda^k V$*

Proof. In order to avoid confusion with the notation we consider the case $k = 2$. Suppose that $\lambda = \lambda_{ij}\boldsymbol{\beta}^i \wedge \boldsymbol{\beta}^j = 0$ with$j > i$ vanishes. Applying this to $(\mathbf{b}_1, \mathbf{b}_2)$ for example, we get that

$$\lambda(\mathbf{b}_1, \mathbf{b}_2) = \lambda_{ij}2!Alt(\boldsymbol{\beta}^i \otimes \boldsymbol{\beta}^j)(\mathbf{b}_1, \mathbf{b}_2) = \lambda_{ij}2!(\frac{1}{2}\boldsymbol{\beta}^i \otimes \boldsymbol{\beta}^j)(\mathbf{b}_1, \mathbf{b}_2) - \frac{1}{2}\boldsymbol{\beta}^i \otimes \boldsymbol{\beta}^j)(\mathbf{b}_2, \mathbf{b}_1))$$

Being $i < j$ only one term survive , namely the first one, proving that $\lambda_{1,2} = \lambda(\mathbf{b}_2, \mathbf{b}_1)$. In this way we prove that all λ_{ij} vanishes proving again linear independence. Do our basis span $\Lambda^k V$? Consider a general $\rho \in \Lambda^2 V$ and define $\rho' := \rho(\mathbf{b}_i, \mathbf{b}_j)\boldsymbol{\beta}^i \wedge \boldsymbol{\beta}^j$ the on every pair $(\mathbf{b}_k, \mathbf{b}_m)$ one finds $\rho = \rho'$ □

Notation It is useful to write any element of $\Lambda^k V$ as

$$\omega = \frac{1}{k!}\omega_{i_1..i_k}\boldsymbol{\beta}^{i_1} \wedge .. \wedge \boldsymbol{\beta}^{i_k}$$

where we are not assuming the ordering of indices, paying the price of the factor $\frac{1}{k!}$ and where

$$\omega_{i_1..i_k} = \omega(\mathbf{b}_{1_1}, .., \mathbf{b}_{i_k})$$

and the symbols $\omega_{i_1..i_k}$ are completely antisymmetric in the sense that we get a minus sign every time we exchange two indices. This is like we were working with upper triangular matrices and we work now with antisymmetric matrices. Tensors are in fact in some sense generalization of matrices while alternating tensors are generalization of antisymmetric matrices. Let's see that in an easy example:

Example Consider $\Lambda^2\mathbb{R}^3$ with canonical basis $\{\mathbf{e}_i\}$ and dual $\{\boldsymbol{\epsilon}^j\}$. According to the previous theorem any element ω of this space can be written as

$$\omega_{12}\boldsymbol{\epsilon}^1 \wedge \boldsymbol{\epsilon}^2 + \omega_{13}\boldsymbol{\epsilon}^1 \wedge \boldsymbol{\epsilon}^3 + \omega_{23}\boldsymbol{\epsilon}^2 \wedge \boldsymbol{\epsilon}^3$$

Defining $\omega_{21} = -\omega_{12}$ and so on, and using the properties of the wedge product we can obviously write the previous expression as

$$\frac{1}{2}\omega_{ij}\boldsymbol{\epsilon}^i \wedge \boldsymbol{\epsilon}^j \ , \qquad i,j = 1,2,3$$

Moreover

$$\omega(\mathbf{e}_1, \mathbf{e}_2) = \omega_{12} 2Alt(\boldsymbol{\epsilon}^1 \otimes \boldsymbol{\epsilon}^2)(\mathbf{e}_1, \mathbf{e}_2) = \omega_{12}$$

We are now really temped to write $T^kV = V^* \otimes .. \otimes V^*$, but what it means? what are the properties of this symbol $\otimes$? In order to better understand it let's go back to our first example $\tau_{\alpha,\beta}$. This is an example of a "decomposable" tensor that can be indeed written as $\alpha \otimes \beta := \alpha_i\beta_j\boldsymbol{\beta}^i \otimes \boldsymbol{\beta}^j$ where in a standard notation we have denoted by α_i the components of the covect α with respect to the basis $\{\boldsymbol{\beta}^i\}$.Observe that not every tensor are decomposable and can be thus be written as the tensor product of vectors and covectors; in particular in the case of $T^2\mathbb{R}^2$, for example, the tensor

$$\boldsymbol{\epsilon}^1 \otimes \boldsymbol{\epsilon}^1 + 2\boldsymbol{\epsilon}^2 \otimes \boldsymbol{\epsilon}^2$$

with $\{\boldsymbol{\epsilon}^1, \boldsymbol{\epsilon}^2\}$ being the canonical base of $(\mathbb{R}^2)^*$, can not be written as $\alpha \otimes \beta$ for some $\alpha, \beta \in (\mathbb{R}^2)^*$ (you must try to convince yourself). Let's now be a tiny bit more abstract. Given a set S one can consider linear formal combination of elements of S, something of the form $\sum a_i x_i$ with a_i real numbers and $x_i \in S$. Formal linear combinations can be understood as a function $f : S \to \mathbb{R}$ vanishing for all but finitely many $x \in S$. More in details denoting by x^* the function vanishing on all elements of S but x and with $x^*(x) = 1$ we have that $f = \sum a_i x_i^*$; idetifying then x and x^* we get the desired definition of the free vector space $\mathbb{R} < S >$. Consider two finite dimensional vectors spaces V and W and construct another vector space, consisting of formal linear combination of objects of the form (α, μ), with $\alpha \in V^*$ and $\mu \in W^*$ viewed as multilinear map from $V \times W$ to the reals. This is then by definition the free vector space $\mathbb{R} < V^* \times W^* >$. We consider the subspace I spanned by all elements of the form

$$a(\alpha, \mu) - (a\alpha, \mu)\,, \quad a(\alpha, \mu) - (\alpha, a\mu)$$

$$(\alpha + \alpha', \mu) - (\alpha, \mu) - (\alpha', \mu)\,, \quad (\alpha, \mu + \mu') - (\alpha, \mu) - (\alpha, \mu')$$

with $a \in \mathbb{R}$. We the define $V^* \otimes W^* := \mathbb{R} < V^* \times W^* > /I$. From this definition we naturally obtain the desired relations discussed in the example 7.2:

$$\begin{aligned} a(\alpha \otimes \mu) &= (a\alpha) \otimes \mu = \alpha \otimes (a\mu) \\ \alpha \otimes \mu + \alpha' \otimes \mu &= (\alpha + \alpha') \otimes \mu \\ \alpha \otimes \mu + \alpha \otimes \mu' &= \alpha \otimes (\mu + \mu') \end{aligned}$$

Proposition 7.7. *Given V and W finite dim vector spaces and S any vector space. Given a BILINEAR map $f : V \times W \to S$ there is a unique LINEAR map $\tilde{f} : V \otimes W \to S$ such that the following diagram commutes:*

$$\begin{array}{ccc} V \times W & \xrightarrow{f} & S \\ \downarrow \pi & \nearrow \tilde{f} & \\ V \otimes W & & \end{array}$$

where $\pi(\mathbf{v}, \mathbf{w}) = \mathbf{v} \otimes \mathbf{w}$

Proof. Sketchy: we first extend f uniquely to a linear map $\bar{f} : \mathbb{R} < V \times W > \to S$ defined by$\bar{f}(\mathbf{v}, \mathbf{w}) = f(\mathbf{v}, \mathbf{w})$ whenever $(\mathbf{v}, \mathbf{w}) \in V \times W \subset \mathbb{R} < V \times W >$. We then note that the subset I defined previously is contained in the kernel of $\bar{f}$ therefore $\bar{f}$ descends to a linear map $\tilde{f} = \mathbb{R} < V \times W > /I \to S$ satisfying $\tilde{f} \circ \pi = f$ by construction. Since every element of $V \otimes W$ can be written as linear combination of object of the form $\mathbf{v} \otimes \mathbf{w}$ and on such elements $\tilde{f}$ is uniquely determined by $\tilde{f}(\mathbf{v} \otimes \mathbf{w}) = \bar{f}(\mathbf{v}, \mathbf{w}) = f(\mathbf{v}, \mathbf{w})$, then uniqueness follows. □

Proposition 7.8. *The vector space $V^* \otimes W^*$ is canonically isomorphic to the vector space $Bil(V, W)$ of bilinear function from $V \times W$ to the reals , thus we will identify them.*

Proof. In order to simplify the notation we consider the case $V = W$. First define the map $f : V^* \times V^* \to Bil(V, V)$ as follows:

$$f(\alpha, \beta)(\mathbf{v}, \mathbf{w}) = \alpha(\mathbf{v})\beta(\mathbf{w})$$

Then by the previous proposition we have a unique $\tilde{f} : V^* \otimes V^* \to Bil(V, V)$. We claim that this is an isomorphism o vector spaces. Consider the basis $\{\mathbf{b}_i\}$ and $\{\boldsymbol{\beta}^i\}$ basis for V and V^*; we know that any element ρ of $V^* \otimes V^*$can be written as $\rho_{ij}\boldsymbol{\beta}^i \otimes \boldsymbol{\beta}^j$. We then define the map $g : Bil(V, V) \to V^* \otimes V^*$ as follows

$$g(B) = B(\mathbf{b}_i, \mathbf{b}_j)\boldsymbol{\beta}^i \otimes \boldsymbol{\beta}^j$$

We now claim that g is the inverse of $\tilde{f}$

$$\begin{aligned} (g \circ \tilde{f})(\rho) &= \tilde{f}(\rho)(\mathbf{b}_i, \mathbf{b}_j)\boldsymbol{\beta}^i \otimes \boldsymbol{\beta}^j \\ \text{by linearity} &= \rho_{km}\tilde{f}(\boldsymbol{\beta}^k \otimes \boldsymbol{\beta}^m)(\mathbf{b}_i, \mathbf{b}_j)\boldsymbol{\beta}^i \otimes \boldsymbol{\beta}^j \\ \text{by construction of } \tilde{f} &= \rho_{km}f(\boldsymbol{\beta}^k, \boldsymbol{\beta}^m)(\mathbf{b}_i, \mathbf{b}_j)\boldsymbol{\beta}^i \otimes \boldsymbol{\beta}^j \\ \text{by definition of } f &= \rho_{km}\delta_i^k\delta_j^m\boldsymbol{\beta}^i \otimes \boldsymbol{\beta}^j = \rho \end{aligned}$$

Along the same line for a general $B \in Bil(V, V)$ and $\mathbf{v}_1, \mathbf{v}_2 \in V$ we have

$$\begin{aligned} \Big((\tilde{f} \circ g)(B)\Big)(\mathbf{v}_1, \mathbf{v}_2) &= B(\mathbf{b}_i, \mathbf{b}_j)\tilde{f}(\boldsymbol{\beta}^i \otimes \boldsymbol{\beta}^k)(\mathbf{v}_1, \mathbf{v}_2) \\ \text{by construction of } \tilde{f} &= B(\mathbf{b}_i, \mathbf{b}_j)f(\boldsymbol{\beta}^i, \boldsymbol{\beta}^j)(\mathbf{v}_1, \mathbf{v}_2) \\ \text{by definition of } f &= B(\mathbf{b}_i, \mathbf{b}_j)v_1^i v_2^j \\ \text{by linearity} &= B(\mathbf{b}_i v_1^i, \mathbf{b}_j v_2^j) = B(\mathbf{v}_1, \mathbf{v}_2) \end{aligned}$$

Note that the construction of g is completely basis independent that's why we cal lit canonical. Being the isom. canonical we will just say that $Bil(V, V) = V^* \otimes V^*$ □

Corollary 7.9. *Let V be a finite dimensional vector space then*

$$T_r^k V = \underbrace{V^* \otimes .. \otimes V^*}_{k \text{ times}} \otimes \underbrace{V \otimes .. \otimes V}_{r \text{ times}}$$

Let us better understand the meaning of those objects with some example.

- $T^0_0 V = \mathbb{R}$
- $T^1 V = V^*$
- $T_1 V = V$
- on $V = \mathbb{R}^n$ $det(\mathbf{v}_1, .., \mathbf{v}_n) \in \Lambda^n V \subset \underbrace{V^* \otimes .. \otimes V^*}_{\text{n times}}$
- a scalar product on V is an element of $\Sigma^2 V \subset V^* \otimes V^*$ of element of the form $g_{ij}\boldsymbol{\beta}^i \otimes \boldsymbol{\beta}^j$ with $g_{ij} = g_{ji}$

What about $End(V)$? We note that there is a canonical isomorphism $\phi : End(V) \to T^1_1 V$ given by $(\phi(L))(\mathbf{v}, \alpha) = \alpha(L(\mathbf{v}))$; this map is injective and since $dimEnd(V) = n^2 = dim(V^* \otimes V^*)$ we get the desired result. In components, given $T^j{}_i \boldsymbol{\beta}^i \otimes \mathbf{b}_j$ we construct the endomorphism $L_T(\mathbf{v}) := T^j{}_i \boldsymbol{\beta}^i(\mathbf{v}) \otimes \mathbf{b}_j$. In analogy with this construction we have

Proposition 7.10. *the vector space T^k_{p+1} is canonically isomorphic to the space of multiliear maps*

$$\underbrace{V \times .. \times V}_{k\ times} \times \underbrace{V^* \times .. \times V^*}_{p\ times} \to V$$

Let us "bundleize" this construction

Definition 7.11. Covariant tensor bundle and covariant tensor field of rank k

$$T^k M := \sqcup_p T^k(T_p M) \;\; \to \;\; \mathcal{T}^k(M) := \Gamma(T^k M)$$

Controvariant tensor bundle and contovariant tensor field of rank r

$$T_r M := \sqcup_p T_r(T_p M) \;\; \to \;\; \mathcal{T}_r(M) := \Gamma(T_r M)$$

Mixed tensor bundle and mixed tensor field of rank (k, r)

$$T^k_r M := \sqcup_p T^k_r(T_p M) \;\; \to \;\; \mathcal{T}^k_r(M) := \Gamma(T^k_r M)$$

In a given local coordinate system with coordinate functions x^i any tensor can be written as

$$T \in \mathcal{T}^k_r(M) \quad \to \text{“locally”} \quad T = T^{j_1..j_r}{}_{i_1..i_k} dx^{i_1} \otimes .. \otimes dx^{i_k} \otimes \partial_{j_1} \otimes .. \otimes \partial_{j_r}$$

where the components $T^{j_1..j_r}{}_{i_1..i_k}$ are smooth functions. As usual we will denote by $T_p \in T^k_r(T_p M)$ the value of a tensor field at a point namely

$$T_p(X_{1_p}, ., X_{k_p}, \omega_{1_p}, .., \omega_{r_p}) = T(X_1, .., X_k, \omega_1, .., \omega_r)(p)$$

Consider now an element S of $\mathcal{T}^1_1(M)$. In a given coordinate chart it looks like $S = S^j{}_i dx^i \otimes \partial_j$. When it acts on a vector $\mathbf{v} = v^i \partial_i$ and $\omega = \omega_i dx^i$, using $\partial_i(dx^j) = \delta^k_i = dx^j(\partial_i)$ we have

$$S(\mathbf{v}, \omega) = S^j_i v^k \omega_m dx^i(\partial_k) dx^m(\partial_j) = S^j_i v^i \omega_j$$

It is kind of evident now that this map is $C^\infty(M)$ linear. This can obviously generalized to any tensor field. So a (1,1) tensor field is a $C^\infty(M)$ multilinear map from $\mathfrak{X}(M) \times \Omega^1(M) \to C^\infty(M)$.There is something more:

Lemma 7.12. *a map*

$$\underbrace{\mathfrak{X}(M) \times .. \times \mathfrak{X}(M)}_{k\ times} \times \underbrace{\Omega^1(M) \times .. \times \Omega^1(M)}_{r\ times} \to C^\infty(M)$$

or

$$\underbrace{\mathfrak{X}(M) \times .. \times \mathfrak{X}(M)}_{k\ times} \times \underbrace{\Omega^1(M) \times .. \times \Omega^1(M)}_{r-1\ times} \to \mathfrak{X}(M)$$

is induced by a rank (k,r) tensor field if and only if it is multilinear over $C^\infty(M)$

Let $\rho \in \mathcal{T}^k(N)$ and $F : M \to N$ a smooth map. We can pullback covariant k tensor in analogy with cavariant vector as follows:

$$(F^*\rho)_p(X_1, .., X_k) = \rho_{F(p)}(F_*X_1, .., F_*X_k)$$

with $X_1, .., X_k$ vector fiedlds on M.

Poperties:

- F^* is linear over $\mathbb{R}$
- $F^*(f\rho) = (f \circ F)F^*\rho$
- $F^*(\rho \otimes \omega) = F^*\rho \otimes F^*\omega$
- $(G \circ F)^* = F^* \circ G^*$

7.1. **Differential forms and integrations.**

Definition 7.13.

$$\Lambda^k M = \sqcup_p \Lambda^k(T_pM)$$

Sections of $\Lambda^k M$ are named differential forms of rank k and the set of those objects is denoted by $\Omega^k(M)$. A differential k form can be written, in a given coordinate chart, as

$$\omega = \frac{1}{k!}\omega_{i_1..1_k} dx^{i_1} \wedge .. \wedge dx^{i_k}$$

Lemma 7.14. *$F : M \to N$ a smooth map, and consider the coordinate functions $(\mathbf{x}) = (x^1, .., x^n)$ for M and $(\mathbf{y}) = (y^1, .., y^m)$ for N (defined locally obviously) then:*

$$F^*(\omega_{i_1..i_k} dy^{i_1} \wedge .. \wedge dx^{y_k}) = (\omega_{i_1..i_k} \circ F) d(y^{i_1} \circ F) \wedge .. \wedge d(y^{i_k} \circ F)$$

Proof. it follows easily from the properties discussed previously and Lemma (6.5) that is $F^*df = d(f \circ F)$. □

This Lemma has a useful corollary:

Corollary 7.15. *Let $F : M \to N$ smooth map between n manifolds then choose the coordinates functions $(\mathbf{x})$ on $U \subset M$ and $(\mathbf{y})$ on $V \subset N$ we have on $U \cap F^{-1}(V)$*

$$F^*(f dy^1 \wedge .. \wedge dy^n) = (f \circ F)\, det(\partial_i F^j) dx^1 \wedge .. \wedge dx^n$$

Proof. Note that

$$\omega^1 \wedge .. \wedge \omega^n(X_1, .., X_n) = det(\omega^i(X_j))$$

as one can easily observe that in the case $n = 3$ (discussed in the exercise session). In particular using the previous Lemma we have

$$F^*(f dy^1 \wedge .. \wedge dy^n) = (f \circ F) d(F^1) \wedge .. \wedge d(F^n)$$

with $dF^i = y^i \circ F$. The components of $d(F^1) \wedge .. \wedge d(F^n)$ are

$$d(F^1) \wedge .. \wedge d(F^n)(\partial_1, .., \partial_n) = det(\partial_i F^j)$$

thus

$$d(F^1) \wedge .. \wedge d(F^n) = det(\partial_i F^j) dx^1 \wedge .. \wedge dx^n$$

□

For any smooth manifold there is a differential operator

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

We will write it in coordinates now, in the following we will show a more general definition.

$$d\omega := \frac{1}{k!}(d\omega_{i_1..i_k}) \wedge dx^{i_1} \wedge .. \wedge dx^{i_k}$$

that can be written as

$$d\omega := \frac{1}{(k+1)!}(\partial_{[j}\omega_{i_1..i_k]})dx^j \wedge dx^{i_1} \wedge .. \wedge dx^{i_k}$$

where $[j, i_1, , i_k]$ means that we are taking the totally alternating part on the indices, that is if we exchange two of them we get a minus sign morally.

Properties of the exterior derivative:

- d is well defined independently of the coordinates chosen (we will see that later with the coordinate independent formula)
- d is linear over $\mathbb{R}$
- $d^2 = 0$
- $\omega \in \Omega^k(M)$ and $\rho \in \Omega^n(M)$ then

$$d(\omega \wedge \rho) = (d\omega) \wedge \rho + (-1)^k \omega \wedge d\rho$$

Let's see an example.Consider on $\mathbb{R}^3$ the one form

$$\omega = f dx + g dy + h dz$$

then

$$d\omega = df \wedge dx + dg \wedge dy + dh \wedge dz = (\partial_x g - \partial_y f)dx \wedge dy + (\partial_x h - \partial_z f)dx \wedge dz + (\partial_y h - \partial_z g)dy \wedge dz$$

where you see the components of the curl of the vector field $f\partial_x + g\partial_y + h\partial_z$ (modulo signs).

Let's go back to vector spaces: Let V be a finite dimensional vector space then an orientation for V is an equivalence class of ordered basis defined to be equivalent if they are related by a positive determinant matrix. A basis is called positively oriented if it belong to the chosen orientation.

Lemma 7.16. *Consider V a real vector space of dimension n and $\Omega \in \Lambda^n V$ then the set of bases $(\mathbf{b}_1, .., \mathbf{b}_n)$ so that $\Omega(\mathbf{b}_1, .., \mathbf{b}_n) > 0$ is an orientation for V*

Proof. Given $(\mathbf{b}_1, .., \mathbf{b}_n)$ (with dual basis $(\boldsymbol{\beta}_1, .., \boldsymbol{\beta}_n)$) and $(\mathbf{b}'_1, .., \mathbf{b}'_n)$ two basis related by

$$\mathbf{b}'_i = A^j_i \mathbf{b}_j$$

and so that $\Omega(\mathbf{b}_1, .., \mathbf{b}_n) = c\boldsymbol{\beta}^1 \wedge .. \wedge \boldsymbol{\beta}^n((\mathbf{b}_1, .., \mathbf{b}_n)) = c > 0$ then

$$\Omega(\mathbf{b}'_1, .., \mathbf{b}'_n) = c \det(\boldsymbol{\beta}^i(\mathbf{b}'_j)) = c \det(A^i_j)$$

then Ω define a class of basis with the same orientation by the previous formula ☐

We will say that given an orientd vector space $\Omega \in \Lambda^n V$ is positively oriented if itinduces the same orientation of V. In the case of a manifold M an orientation is a choice of orientation for each T_pM that we want to be smooth in the sense that in a neighborhood of point we cna always find a local frame pointwise positively oriented. A coordinate chart (U, φ) is said to be positive oriented if the coordinate basis $\{\frac{\partial}{\partial x_i}|_p\}$ for T_pM is positively oriented for each $p \in U$.

Proposition 7.17. *Let M be a smooth manifold of dimension n. A nowhere vanishing $\Omega \in \Omega^n(M)$ (sometimes called volume form) determines a unique orientation of M for which Ω is positively oriented at each point.*

Proof. In a coordinate chart (U, φ) with U connected, we can write $\Omega = f dx^1 \wedge .. \wedge dx^n$ with f nowhere vanishing and we have

$$\Omega(\partial_1, .., \partial_n) = f$$

thus on U it is alway positive or negative oriented. If it's negative oriented it is enough to change the sign of one of the coordinate representation of any point and obtain thus a continuous orientation. □

One can also prove that conversely an orientation for M induce a nonvanishing n form positively oriented at each point. Consider now a top form defined on a compact domain $D \subset \mathbb{R}^n$

$$\omega = f dx^1 \wedge .. \wedge dx^n$$

and we define

$$\int_D \omega = \int_D f dx^1..dx^n$$

Given an open U we can always find a compact integration domain D s.t. $\text{supp}(f) \subset D \subset U$ thus we define

$$\int_U \omega := \int_D \omega = \int_D f dx^1..dx^n$$

where againsupp$(f) \subset D \subset U$

Proposition 7.18. *Suppose U, V are open subsets of $\mathbb{R}^n$, $F : V \to U$ is an orientation-preserving diffeomorphism, then*

$$\int_U \omega = \int_V F^* \omega$$

Proof. This is just a consequence of 7.15 and the change of coordinates for an integral (absolute value of the determinant of the Jacobian). □

We are ready to consistently define the integral on a manifold M. Given ω compactly supported in an oriented coordinate chart (U, φ)

$$\int_M \omega := \int_{\varphi(U)} (\varphi^{-1})^* \omega$$

By construction this definition does not depend on the choice of oriented charts whose domain contain suppω. If the n form is supported on M oriented we can use the partition of unity $\{\psi_i\}$ subordinate to a coordinate charts (U_i, φ_i) covering M considering on each U_i the one form $\psi_i \omega$ supported there, and then summing overall i

$$\int_M \omega := \sum_i \int_M \psi_i \omega$$

Note: One can prove that the final answer does NOT depend on the choice of coordinates charts and the partition of unity.

8. Integral curves and symmetry generators

Definition 8.1. Given $X \in \mathfrak{X}(M)$, an integral curve of X is a curve $\gamma : I \to M$ on some open interval $I \subseteq \mathbb{R}$ such that

$$X(\gamma(t_0)) := X_{\gamma(t_0)} = \dot{\gamma}(t_0)$$

We will sometimes say that the curve starts at p_0, or that p_0 is the initial point for γ, whenever $p = \gamma(t_0)$ for some given t_0; if t_0 is not specified we will often assume $t_0 = 0$. Morally an integral curve is a solution of a systems of ODEs. In coordinates, suppose we have a chart (U, φ) we can choose $J \subseteq I$ so that $\gamma(J) \subseteq U$, then we have

$$\dot{\gamma}(t) = \gamma^i(t)\partial_i$$

Along the curve we have $X_{\gamma(t)} = a^i(\gamma(t))\partial_i|_{\gamma(t)}$ on U thus we can conclude that

$$\dot{\gamma}^i(t) = \hat{a}(\gamma^1(t), .., \gamma^n(t)), \qquad \forall t \in J \tag{1}$$

Theorem 8.2. *Consider thie previous ODE we have:*

- Unique: *every 2 solutions $\gamma_1 : I_1 \to U$ and $\gamma_2 : I_2 \to U$ s.t. $t_0 \in I_1 \cap I_2$ and $\gamma_1(t_0) = \gamma_2(t_0)$ agree on $I_1 \cap I_2$*
- Existence: *Fore very t_0, p in I and U and $V \subseteq U$ neighborhood of p, then exist on V a curve satisfying the previous ODE with initial condition $\gamma(t_0) = p$*
- Smoothness: *The solution of the previous point depends smoothly on the choice of the initial point and t.*

Consider now the set of integral curve of X starting @ p, and define a partial order by $\gamma_1 \leq \gamma_2$ whenever $I_1 \subseteq I_2$

Definition 8.3. A maximal integral curve of X starting @ p is a solution of equation (1) that is greater or equal to other solutions (or if you prefer is not contained in other solutions). We will denote it by

$$\gamma_p : I_p \to M$$

and I_p is named the maximal interval.

A vector field X on a manifold is name complete if every maximal integral curve of X is defined on the whole $\mathbb{R}$, i.e. $I_p = \mathbb{R}$ for all p.

Proposition 8.4. *The maximal curve of X starting @ p is unique*

Given the solution discussed above we can imagine that the initial point $p \in M$ moves along the maximal integral curve in the sense that after a time t it reaches the point $\gamma_p(t)$. This machine is called Flow of the vector field. More in details

Definition 8.5. Given X and

$$D := \{(t, p) \in \mathbb{R} \times M : t \in I_p\} \subseteq \mathbb{R} \times M$$

named the flow domain, the flow of X is the map

$$Fl^X : D \to M$$

defined by

$$Fl^X(t, p) := Fl^X_t(p) = \gamma_p(t)$$

Theorem 8.6. *We have:*

- *Fl^X is a smooth map*
- $(Fl^X_t \circ Fl^X_s)(p) = (Fl^X_{t+s})(p)$

- $Fl^X_0 = id_M$

The vector field X is called the infinitesimal generator of the flow and it is strictly related to the concept of symmetries of a manifold. Let's see that with examples:

Example 8.7.
$$X = x\partial_x + y\partial_y \in \mathfrak{X}(\mathbb{R}^2)$$
then
$$Fl^X_t(x_0, y_0) = (e^t x_0, e^t y_0)$$
and we will say that it generates the dilations

Example 8.8.
$$X = \partial_x \in \mathfrak{X}(\mathbb{R}^2)$$
then
$$Fl^X_t(x_0, y_0) = (x_0 + t, y_0)$$
and we will say that ∂_x generates the translation along the x axis.

Observation: Fixed $t \in \mathbb{R}$ we can define
$$M_t = (p \in M : (t, p) \in D)$$
notice it can be empty. Then there is a well defined map
$$\Phi_t : M_t \to M$$
The map Φ_t induced by a vector field X is sometimes called the local 1 parameter group of local diffeomorphisms generated by X, and X is called the infinitesimal generator. This terminology is motivated by (8.6).

Definition 8.9. Given $f \in C^\infty(M)$ and X and Y vector fields we have

- $(\mathcal{L}_X f)(p) = \partial_t|_0(f \circ Fl^X_t(p))$
- $(\mathcal{L}_X Y)(p) = \partial_t|_0((Fl^X_{-t})_* Y_{Fl^X_t(p)})$ the Lie derivative of a function and a vector field respectively.

In order to better understand the Lie bracket of a vector field we take a small detour and discuss a famous algebraic object. Given two vectors fields X and Y we can induce a third one, we denote by $[X, Y]$, by the following algebraic operation called Lie bracket (or sometimes commutator) defined at a point by:
$$[X, Y]_p f = X_p \cdot (Y \cdot f) - Y_p \cdot (X \cdot f)$$

Proposition 8.10. *We have:*

- $[X, Y]_p \in T_pM$
- *The assignment* $p \to [X, Y]_p$ *defines a smooth vector field* $[X, Y]$ *satisfying* $[X, Y]f = X \cdot Y \cdot f - Y \cdot X \cdot f$
- *In coordinates given* $X = a^i\partial_i$ *and* $Y = b^j\partial_j$ *we have*
$$[X, Y] = (a^j\partial_j b^i - b^j\partial_j a^i)\partial_i$$

It is useful to think at those square brackets as an algebraic operator called Lie bracket
$$[\,,\,] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$
such that

- $[\,,\,]$ is $\mathbb{R}$ bilinear
- is skew symmetric

- satisfies the so called Jacobi identity, namely

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

Proposition 8.11.

$$(\mathcal{L}_X f)(p) = \partial_t|_0(f \circ Fl_t^X(p)) = X_p \cdot f$$

and

$$(\mathcal{L}_X Y)(p) = [X, Y]$$

Proof. (*) Let's see the first relation

$$(\mathcal{L}_X f)(p) = \partial_t|_0(f \circ Fl_t^X(p)) = \partial_t|_0 f(\gamma_p(t)) = \dot{\gamma}(0) \cdot f = X_p \cdot f$$

The second relation is more involved, let s see it step by step. In order to simplify the notation we denote by $F(t, p) = Fl_t^X(p)$ and F^i the coordinates representation of this function in some coordinates chart and $X = a^i\partial_i$ and $Y = b^j\partial_j$. We first observe that by definition

$$F^i(t, p) = \gamma_p^i(t)$$

thus $\dot{F}^i(0, p) = a^i(p)$ while

$$\partial_k|F^i(0, p) = \delta_k^i$$

because $F^i(0, p) = \varphi(\gamma_p^i(0)) = \varphi(p) = x^i$. Then

$$(Fl_{-t}^X)_* Y_{Fl_t^X(p)} = (Fl_{-t}^X)_*(b^i(F(t, p))\partial_i|_{F(t,p)}) = b^i(F(t, p))\partial_i F^j(-t, F(t, p))\partial_j|_p$$

We take now the derviative with respect tot and using the relation described before we get

$$\begin{aligned}
\partial_t|_{t=0}(Fl_{-t}^X)_* Y_{Fl_t^X(p)} &= (\partial_t F^k(0, p)\partial_k b^i(p)\partial_i F^j(0, p) - b^i(p)\partial_t\partial_i F^j(0, p) \\
&\quad + b^i(p)\partial_t F^k(0, p)\partial_k\partial_i F^j(0, p))\,\partial_j|_p \\
&= \partial_j b^i|_p a^j(p)\partial_i|_p - \partial_j a^i|_p b^j(p)\partial_i|_p = ([X, Y])_p^i\partial_i|_p
\end{aligned}$$

where in the last line we used that $\partial_i F^k = \delta_i^k$ thus $\partial_m\partial_i F^k = 0$. □

Let's see how vector fields can be efficiently used.

Definition 8.12. A symmetry of a smooth function f on a manifold M is a diffeomorphism preserving f. We will call it local if it applies on a submanifold on M only.[3]

Let's see an example. Consider on $\mathbb{R}^2$ a function depending on $r^2 = x^2 + y^2$ only, that is $f(x, y) = f(r(x, y))$. Consider the the rotation map $R(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$. Then one has that $(R^* f)(r(x, y)) = f(r(R(x, y)) = f(r(x, y))$

Definition 8.13. An infinitesimal symmetry of a smooth function f is a vector field $X \in \mathfrak{X}(M)$ whose flow (for fixed t thus Φ_t) is a local symmetry for f.

Consider on $\mathbb{R}^2$ for example $f = f(y)$, we have seen that the vector field ∂_x generates translation along the x-axis,that this is a symmetry for f.

Can we define symmetries for a vector field too??Sure

Definition 8.14. A symmetry of a vector field Y on a manifold M is a diffeomorphism $\Phi : M \to M$ preserving Y in the sense that $\Phi_* Y = Y$. A local symmetry of Y is a diffeomorphism between open submanifolds of M .

[3] A submaifold is a subset with a smooth structure induced by M; we will define it more carefully in the next chapter

Given for example on $\mathbb{R}^2$ the vector field $X = x\partial_x + y\partial_y$ then the rotation is a symmetry. In fact we have

$$R_*(x\partial_x + y\partial_y)f(x,y) = (x\partial_x + y\partial_y)f(z,t)$$

with $z = x\cos\theta - y\sin\theta$ and $t = x\sin\theta + y\cos\theta$. then by the chain rule we get $R_*X = t\partial_t + z\partial_z$. In general we can play the following game. $R_*X = A\partial_x + B\partial_y$; by construction we have that $A = (R_*X)\cdot x = X\cdot(x\circ R)$

Definition 8.15. An infinitesimal symmetry of a vector field Y is another vector field X on whose flow (for fixed t thus Φ_t) induce a local symmetries of Y.

Proposition 8.16. *X is a local symmetry for f iff $X\cdot f = 0$; X is a local symmetry for Y iff $[X,Y] = 0$*

Proof. If X is a local symmetry then $(Fl_t^X)^*f(p) = f(Fl_t^X(p)) = f(p)$ then f is constant along the flow and

$$(\mathcal{L}_X f)(p) = (X\cdot f)(p) = \partial_t|_{t=0}(Fl_t^X)^*f = 0$$

while if $X\cdot f = 0$ we have that $\partial_t(Fl_t^X)^*f = 0$ then $f(Fl_t^X(p))$ is constant along the intergral curve for X. Similarly one gets the other statement. □

The Lie derivative can be extended to any covariant tensor field τ as follows:

$$(\mathcal{L}_X\tau)(p) = \partial_t|_0((Fl_t^X)^*\tau_{Fl_t^X(p)})$$

Proposition 8.17. *Let σ and τ be covariant tensors f a smooth function and X a vector field then*

- $\mathcal{L}_X(f\sigma) = (\mathcal{L}_X f)\sigma + f\mathcal{L}_X(\sigma)$
- $\mathcal{L}_X(\sigma\otimes\tau) = \mathcal{L}_X(\sigma)\otimes\tau + \sigma\otimes\mathcal{L}_X(\tau)$
- $\mathcal{L}_X(\tau(Y_1,..,Y_k)) = \mathcal{L}_X(\tau)(Y_1,..,Y_k) + \tau(\mathcal{L}_X Y_1, Y_2,..,Y_k) + ... + \tau(Y_1,..\mathcal{L}_X Y_k)$

and in particular for a covariant rank k tensor one has:

$$(\mathcal{L}_X\tau)(Y_1,..,Y_k) = X\cdot\tau(Y_1,..,Y_k) - \tau(\mathcal{L}_X Y_1,..,Y_k) - .. - \tau(Y_1,..,\mathcal{L}_X Y_k)$$

We are now ready to go back to the exterior derivative and write it down in a coordinate independent way. Let 's start with the case of a one form α.

$$d\alpha(X,Y) := X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])$$

The first thing to observe (and prove) is that $d\alpha$ is a tensor thus a $C^\infty(M)$ multilinear map. Let's compute its components in a coordinate chart

$$d\alpha = \frac{1}{2}a_{ij}dx^i\wedge dx^j$$

then

$$d\alpha(\partial_i,\partial_j) = a_{ij} = \partial_i\alpha(\partial_j) - \partial_i\alpha(\partial_j) = \partial_i\alpha_j - \partial_j\alpha_i$$

In general we have

$$\begin{aligned} d\alpha(X_1,..,X_{k+1}) \;:=\; & \textstyle\sum_i(-1)^{i+1}X_i\alpha(X_1,..X_{i-1},X_{i+1},..,X_{k+1}) \\ & +\textstyle\sum_{i<j}(-1)^{i+j}\alpha([X_i,X_j],X_1,..X_{i-1},X_{i+1},..,X_{j-1},X_{j+1},..,X_k) \end{aligned}$$

Another way to think at the Lie bracket is the fact that it represents the obstruction to the commutation of the flow of 2 vector fields. Before going into the details of this statement we first point out the following result:

Proposition 8.18. *Given the smooth map $F : M \to N$ and a curve $\gamma : I \to M$ on M, for every $t_0 \in I$ we have that*

$$F_* \dot{\gamma}|_{t_0} = \frac{d}{dt}(F \circ \gamma)|_{t_0}$$

More in details we have the following result

Proposition 8.19. *Two vector fields commutes if and only if so do their flows*

Proof. To avoid any technical complexity we consider the case of complete vector fields, in this case the flows commute means that we have $Fl^Y_s \circ Fl^X_t = Fl^X_t \circ Fl^Y_s$. Assume X and Y commute. Then Fl^X_t is a symmetry for Y for all t. The previous proposition then implies that if $\gamma_p(s) = Fl^Y_s(p)$ is the max integral curve for Y starting at p, so is also $Fl^X_t \circ \gamma_p(s) = Fl^X_t \circ Fl^Y_s(p)$ starting at $Fl^X_t(p)$. By uniqueness it must coincide with $\gamma_{Fl^X_t(p)}$ that is $Fl^Y_s \circ Fl^X_t(p)$.

Now assume that the flows commute. Consider $Y(p)0Y_p$ applying the pushforward given bu the flow of X we get:

$$\begin{aligned} (Fl^X_t)_* Y_p &= (Fl^X_t)_*(\tfrac{d}{ds}|_{s=0} Fl^Y_t(p)) \\ \text{using the previous proposition} &= \tfrac{d}{ds}|_{s=0}((Fl^X_t) \circ (Fl^Y_s))(p) \\ \text{the flows commute by hp} &= \tfrac{d}{ds}|_{s=0}((Fl^Y_s) \circ (Fl^X_t))(p) \\ \text{by definition} &= Y_{Fl^X_t(p)} \end{aligned}$$

Thus X is an infinitesimal symmetry for Y thus they commute □

More in general it implies that if $[X, Y] = 0$ then

$$(Fl^Y_{-t}) \circ (Fl^X_{-t}) \circ (Fl^Y_t) \circ (Fl^X_t)(p) = p$$

but when the vector field do not commute one can prove that

$$(Fl^Y_{-t}) \circ (Fl^X_{-t}) \circ (Fl^Y_t) \circ (Fl^X_t) = Fl^{[X,Y]}_{t^2} + o(t^3)$$

8.1. **Submanifolds.**

Definition 8.20. A subset S of a manifold M (of dimension n) is called a (regular) submanifold of dimension k if for every $p \in S$ we can find a coordinate chart (U, φ) in the maximal Atlas, with $p \in U$ and coordinate functions $(x^1, .., x^n)$ such that $U \cap S$ (better to say $\widehat{U \cap S}$) is defined by the vanishing of $n - k$ coordinates functions that we can assume to be the last $n - k$ coordinates functions without loosing generalities.

We call (U, φ) adapted relative to S. Note that $\varphi|_{U \cap S} = (x^1, .., x^k, 0, .., 0)$ and we can define using the projection on the first k components $pr_{(k)}$

$$\varphi_S := pr_{(k)} \circ \varphi : U \cap S \to \mathbb{R}^k$$

so that $(U \cap S, \varphi_S)$ is a coordinate chart for S with the subspace topology.

Proposition 8.21. *Let S be a regular submanifold of N and (U, φ) a collection of compatible adapted charts of N that covers S. Then $(U \cap S, \varphi_S)$ is an atlas for S. Therefore, a regular submanifold is itself a manifold.*

Submanifolds are typically presented as images or level sets of smooth maps. A level set of a map $F : M \to N$ is a subset

$$F^{-1}(q) := \{p \in M s.t. F(p) = q\}$$

for some $q \in N$. In the case $N = \mathbb{R}^n$ we call $\xi(F) = F^{-1}(\mathbf{0})$ the zero set of F.

Definition 8.22. Given $F : M \to N$ we will say that $q \in N$ is a regular value of F if either q is not in $Im(F)$ or for every $p \in F^{-1}(q)$ we have $F_*|_p : T_pM \to T_{F(p)}N$ is surjective. The preimage of regular value is called a regular level set.

Before we proceed with our analysis let's state a couple of natural results that is the inverse function theorem and its generalization to case of manifolds:

Theorem 8.23. *Let W be an open subset of $\mathbb{R}^n$ and $F : W \to \mathbb{R}^n$ a smooth map. For any point p in W the map F is locally invertible at p if and only if the Jacobian determinant $\partial_i F^j(p)$ is not zero.*

that can be generalized for manifolds as

Theorem 8.24. *Let M be an n dimensional manifold, p a point in M, and U a neighborhood of p. Let $F : U \to \mathbb{R}^n$ be a smooth map. Suppose that in chart (V, ψ) containing p and with copordinate $x^1, .., x^n$, the Jacobian determinant $\partial_i \hat{F}^j(p)$ is non vanishing. Then there is a neighborhood W of p on which F is a diffeomorphism onto its image. Moreover (W, F) is a chart in the max Atlas of M.*

This theorem is usually successfully applied in the following form

Corollary 8.25. *Take a manifold M of dimension n. A set of smooth functions $F^1, .., F^n$ defined on a coordinate chart (U, φ) with coordinates functions $(x^1 .., x^n)$ around a point p, induces a coordinate chart around p if $det \frac{\partial F^i}{\partial x^j}|_p = det \frac{\partial \hat{F}^i}{\partial x^j}|_{\varphi(p)}$ is nonvanishing .*

Proof. Sketchy: we can define $\Phi := (F^1, .., F^n) : U \to \mathbb{R}^n$ by the inverse function theorem it has nonvanishing Jacobian determinant then we can find a neighborhood W of p such that $\Phi : W \to F(W)$ is a diffeomorphism (essentially by local invertibility) then (W, Φ) is a coordinate chart □

Let's discuss an important case

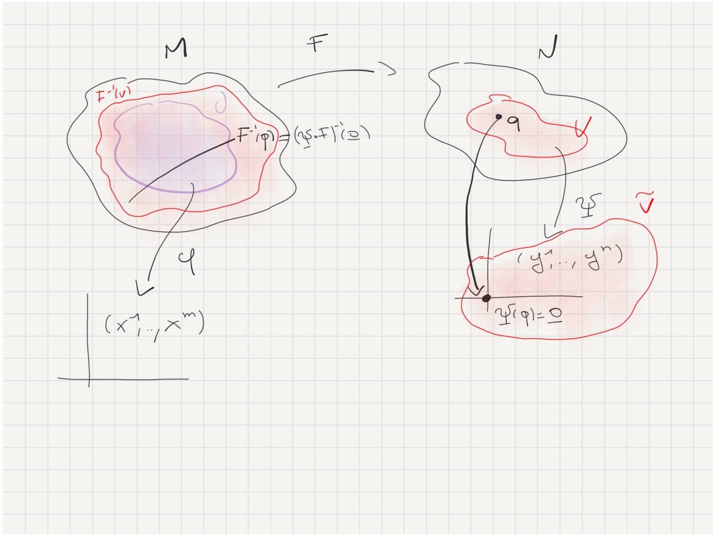
The 2-sphere in $\mathbb{R}^3$: the 2 sphere is the level set $f^{-1}(0)$ of the function $f(x, y, z) = x^2 + y^2 + z^2 - 1$. Note that

$$\partial_x f = 2x, \partial_y f = 2y, \partial_z f = 2z$$

thus for every point p on $S^2 = f^{-1}(0)$

$$T_p\mathbb{R}^3 \to T_0\mathbb{R}$$

given by $f_*(a\partial_x + b\partial_y + c\partial_z) = 2xa + 2by + 2cz$ is surjective because the point $(0, 0, 0)$ (that is the only critical one) does not belong to the sphere then 0 is a regular value. Consider a point $p \in S^2$ such that $\partial_z f|_p$ is nonzero. Then construct the set of functions $\Phi_1 = (x, y, f) : \mathbb{R}^3 \to \mathbb{R}^3$ then the Jacobian is non vanishing on p. For the inverse function theorem there is a neighborhood U_1 of p s.t. (U_1, Φ_1) is a chart for $\mathbb{R}^3$ in the max Atlas and the set $U_1 \cap S^2$ is such that $(U_1 \cap S^2, pr_{(2)}\Phi_1)$ is a coordinate chart for S^2. Along the same line one can prove the same for x and y and construct adapted charts covering S^2 proving that S^2 is a (regular) submanifold.
With this idea in mind one can prove the following:



Theorem 8.26. *Let $F : M \to N$ be a smooth map of manifolds, with dimensions m and n respectively. Then a non empty regular level set, is a regular submanifold of dimension $m - n$.*

Proof. (*) Consider the charts (U, φ) with coordinates functions $x^a = x^1, .., x^m$ on M and (V, ψ) with coordinate functions $(y^i) = y^1, .., y^n$ and centered at a point q with $F^{-1}(V)$ containing U and $F^{-1}(q)$. Observe that o that we can view $F^{-1}(q)$ as the zero set of $\psi \circ F$ since $\psi(q) = \mathbf{0}$. Note now that since the regular level set is assumed to be non empty it means that the differential map at $F^{-1}(q)$ is a surjection then we must have $m \geq n$. Call now $\psi \circ F = F^1, .., F^n$ with F^i smooth functions on M: Note that $F^i(p) = 0$ for every $p \in F^{-1}(q)$. By regularity me must then have that every $p \in F^{-1}(q)$ are such that the jacobian matrix $\partial_a F^j|_p$ has rank n. Without loosing of generality we assume last $n \times n$ block of the Jacobian is non singular (we will denote it by $\partial_{\hat{a}} F^i|_p$). Then we use $F^1, .., F^n$ as the last coordinates. In particular we claim that we have a neighborhood U_p of a fixed $p \in F^{-1}(q)$ such that (U_p, φ^F) is a chart, with $\varphi^F(r) = (x^1, ..., x^{m-n}, F^1, .., F^m)$ for some $r \in U_p$. This chart is well defined because of the inverse function theorem and it is an adapted coordinate chart for $F^{-1}(q)$.

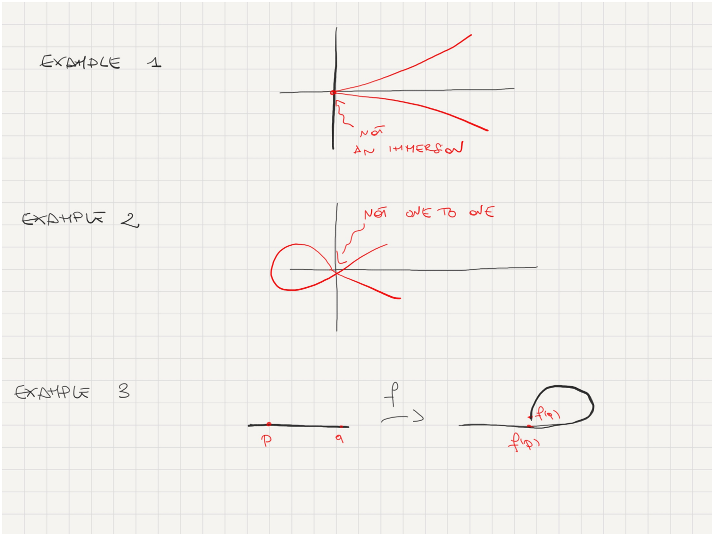
$$\partial_a F^i|_p = \begin{pmatrix} I & 0 \\ 0 & \partial_{\hat{a}} F^i|_p \end{pmatrix}$$

where $\hat{a} = m + 1, .., n$ □

Let us now characterize regular submanifolds.

Definition 8.27. Let the $F : M \to N$ with dimensions m and n. The rank of F at p is the rank of the pushforward at a point that is $rankF_*|_p$ that is $dimIm(F_*|_p)$.

- If $m \leq n$ and for every $p \in M$ we have $rankF = m$, we say that F is an immersion
- If $m \geq n$ and for every $p \in M$ we have $rankF = n$, we say that F is an submersion
- If $m = n$ and for every $p \in M$ we have $rankF = n = m$, we say that F is an local diffeomorphism



Theorem 8.28. *Let $F : M \to N$ a smooth map. Suppose F has constant rank k in a neighborhood of $p \in M$. Then we can find a coordinate chart (U, φ) centered at p with coordinate $(x^1, ..., x^n)$ and (V, ψ) centered at $F(p)$ so that $\hat{F}(x^1, .., x^m) = (x^1, .., x^k, 0, .., 0)$*

This theorem called the constant rank theorem has nice consequences for example if $f : M \to N$ has constant rank in a neighborhood of a level set, then the level set is a regular submanifold.

Consider now a one to one immersion $F : N \to M$; the image $F(N)$ is called immersed submanifold. In general its topology and smooth structure has nothing to do with the one on M and has to be considered as extra data. This observation leads to the following definition:

Definition 8.29. A map $F : M \to N$ is called an embedding if it is a one-to-one immersion and $f(M)$ with the subspace topology is homeomorphic to M trough f.

Let's see some example of non embedding to understand which type of situation we want to avoid:
Example 1 $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = (t^2, t^3)$.This map is one to one but not an immersion since the differential map at zero $f_*|_0$ is not injective.

Example 2 $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = (t^2 - 1, t^3 - t)$.This map is NOT one to one because $f(1) = f(-1)$ and an immersion since the differential map at zero $f_*|_0$ is always injective.

Example 3 Consider the map given in the picture. It is a one to one immersion but the topology induced on $\mathbb{R}^2$ doesn t match the original topology because, for example, there are point close to $f(p)$ corresponding in $\mathbb{R}$ to point far away from p. Thus M and $f(M)$ in this example are not homeomorphic thus it is not an embedding.

Theorem 8.30. *If $F : M \to N$ is an embedding then $F(M)$ is a regular submanifold.*

Proof. (*) By the constant rank theorem we know we can choose local coordinates charts (U, φ) and (V, ψ) such that

$$\hat{F}(x^1, .., x^m) = (x^1, .., x^m, 0, .., 0)$$

We may have trouble since $V \cap F(M)$ may be larger then $F(U)$. We have to find in some neighborhood $F(p)$ in V that $F(M)$ is given by the vanishing of $n-m$ coordinate. It is here where we use the subspace topology to find a V' such that $V' \cap F(N) = F(U)$ (we skip this details) and thus on $V \cap V'$ we can construct the adapted coordinate chart as before. □

9. A mini course on linear connections

Big problem: Consider a section σ of a vector bundle $E \to M$ and we want to study how it changes for example when we move along a curve γ on M starting at p (kind of useful approach). We are tempted to write something like

$$\partial_t \sigma(\gamma(t))|_{t=0} = \lim_{t\to 0} \frac{\sigma(\gamma(t)) - \sigma(p)}{t}$$

This formula in general doesn t make sense; note in fact that given two different points p and q with $\sigma(p) \in E_p$ and $\sigma(q) \in E_q$ BUT E_p and E_q are different vector spaces (even if isomorphic) thus to compare them we must specify the isomorphism $E_p \xrightarrow{\sim} E_q$ that in general is not canonical, unless the bundle is trivial that is $E = M \times \mathbb{R}^k$.

Goal: We now look for and in some sense characterize the isomorphism $P^t_\gamma : E_p \xrightarrow{\sim} E_q$ where $p = \gamma(0)$ and $p = \gamma(t)$ Let's start constructing the following subvector space of E_p

Definition 9.1.

$$V_P E := \{\mathbf{A} \in T_P E \ s.t. \ \pi_* \mathbf{A} = \mathbf{0}\}$$

called vertical vector space at $P \in E$ over $p \in M$ (that is $\pi(P) = p$)

$$VE = \sqcup_{P\in E} V_P E$$

is called the vertical bundle.

Note that $V_P E = T_P E_p$. Sketchy this can be proved as follows. If we take any curve $\gamma(t) \in E_p$ then we have by proposition 8.18 that $\pi_* \dot\gamma(0) = \frac{d}{dt}|_{t=0}(\pi \circ \gamma(t)) = \frac{d}{dt}|_{t=0} p = 0$ proving thus that $T_P E_p \subseteq V_P E$. The other implication can be easily obtained working in a given local trivialization ϕ and using the fact that $\pi = pr_1 \circ \phi$. Taking the pushforward of this "commutative diagram" yields the statement (details are left as an exercise).

We now choose a complement for $V_P E$, we call it the horizontal vector space $H_P E$ so that :

$$T_P E = V_P E \oplus H_P E$$

The choice of $H_P E$ is NOT canonical, it is a CHOICE. We will call $HE = \sqcup_{P\in E} H_P E$ the horizontal bundle. Observe then that $\pi_* : T_P E \to T_p M$ has $V_P E$ as kernel thus it induces an isomorphism

$$Hor_P : T_p M \xrightarrow{\sim} H_P E \subset T_P E$$

Called sometimes the horizontal lift. A curve through the total space E is called horizontal if its tangent vector is horizontal. Then given $p \in M$ and $P \in E_p$, any curve $\gamma(t)$ with $\gamma(0) = p$ lifts uniquely to a horizontal curve $\tilde\gamma(t)$ with $\tilde\gamma(0) = P$ that is

(1) $\pi(\tilde\gamma_P) = \gamma_p$
(2) $\tilde\gamma_P(0) = P$

It can be proven easily that $\dot{\tilde{\gamma}}|_{\tilde{\gamma}(t)} = Hor_{\tilde{\gamma}(t)}\dot{\gamma}|_{\gamma(t)}$. Take now a vector field X on M and $X(p) = X_p \in T_pM$ and define then $\tilde{X}_P := Hor_p(X_p)$ the horizontal lift of X_P at P. Take then γ_p the max integral curve for X at p and construct its lift at $P \in E$ (remember $\pi(P) = p$) called $\tilde{\gamma}_P$ by:

(1) $\pi(\tilde{\gamma}_P) = \gamma_p$
(2) $\tilde{\gamma}_P(0) = P$
(3) $\dot{\tilde{\gamma}}_P|_{\tilde{\gamma}_P(t)} = Hor_{\tilde{\gamma}_P(t)}(X_p) \in H_{\tilde{\gamma}_P(t)}E$

IDEA!!! The isomorphism P_t^γ we are looking form can be constructed out of the flow of the horizontal lift of X

$$Fl_t^{\tilde{X}} : E_p \to E_{\gamma_p(t)}$$

where we have denoted by $\tilde{X}$ the unique vector field on E such that $\tilde{X}(P) = \tilde{X}_P = Hor_P(X_p)$. Observe that by construction the flow is invertible but in principle is not guaranteed that it induce a linear map (and thus a vect space isomorphism). To this aim note that every point on E_p is a vector and it is useful to denote very point P on E_p by the pair $(p, \boldsymbol{\rho})$ or simply $\boldsymbol{\rho}$ when is not crucial to specify that it is a point on the fiber over p. Given $X_p \in T_pM$ It is natural to require that the horizontal map satisfies the following:

$$Hor_{a\boldsymbol{\rho}}(X_p) = aHor_{\boldsymbol{\rho}}(X_p)$$

We will call an horizontal lift satisfyibng the porevious relation linear.

Proposition 9.2. *Given the horizontal lift satisfying the previous relation one has that*

$$Fl_t^{\tilde{X}}(a\boldsymbol{\rho}) = aFl_t^{\tilde{X}}(\boldsymbol{\rho})$$

Proof. Consider $\tilde{\gamma}_{\boldsymbol{\rho}}$ and $\tilde{\gamma}_{a\boldsymbol{\rho}}$ then one has

$$\frac{d}{dt}|_{t=0}(a\tilde{\gamma}_{\boldsymbol{\rho}}) = aHor_{\boldsymbol{\rho}}(X_p) = Hor_{a\boldsymbol{\rho}}(X_p) = \frac{d}{dt}|_{t=0}(\tilde{\gamma}_{a\boldsymbol{\rho}})$$

being the initianl points and the tangent vectors the same the flows must coincide as stated in the proposition. ☐

We use now this observation within the context of the following proposition

Proposition 9.3. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ with $F(a\mathbf{x}) = aF(\mathbf{x})$ then F is linear.*

Proof. The proof relies on the taylor expansion

$$F(\mathbf{x} + \mathbf{x}_0) = F(\mathbf{x}_0) + \partial_{\mathbf{x}}F|_{\mathbf{x}_0}\mathbf{x} + R(\mathbf{x})$$

where the rest R is such that $\frac{|R(\mathbf{x})|}{|\mathbf{x}|} \to 0$ when $x \to 0$.Now we must have $R(a\mathbf{x}) = aR(\boldsymbol{x})$ thus

$$\lim_{a\to 0}\frac{|R(a\mathbf{x})|}{|a\mathbf{x}|} = \frac{|R(\mathbf{x})|}{|\mathbf{x}|}$$

thus $R(\mathbf{x}) = 0$ ☐

We then have the desired result that is the flow induces a linear isomorphism among fibers. We have then all ingredients needed to study how a section σ changes along the integral curve for a vector fields X:

$$(\nabla_X\sigma)(p) := \left.\frac{d}{dt}\right|_{t=0}(Fl_{-t}^{\tilde{X}} \circ \sigma \circ Fl_t^X(p))$$

Computing explicitly the time derivative and identifying $TE_p$with E_p one can easily prove that

(2) $$(\nabla_X \sigma)(p) = -\tilde{X}_{\sigma(p)} + \sigma_* X_p$$

This operator is often called covariant derivative.

OBSERVATION: $\pi_* \tilde{X}_{\sigma(p)} = X_p$ by construction and$\pi_* \sigma_* X_p = (\pi \circ \sigma)_* X_p = X_p$thus the right hand side of the previous relations an element of $V_{\sigma(p)}E$ that we identify with E_p, then we have; taking into account the smoothness of all the ingredients this observation implies that ∇ can be viewed as a map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$$

Proposition 9.4. *We have:*

(1) $\nabla_{fX}\sigma = f\nabla_X \sigma$
(2) $\nabla_{X_1+X_2}\sigma = \nabla_{X_1}\sigma + \nabla_{X_2}\sigma$
(3) $\nabla_X(f\sigma) = f\nabla_X\sigma + (X \cdot f)\sigma$
(4) $\nabla_X(\sigma_1 + \sigma_2) = \nabla_X\sigma_1 + \nabla_X\sigma_2$

Proof.

(1) Let's prove the first relation. Using equation (2) and the linearity of the Hor_P map one has

$$\begin{aligned}(\nabla_{fX}\sigma)(p) &= -\widetilde{(fX)}_{\sigma(p)} + \sigma_*(f(p)X_p) \\ &= -f(p)\tilde{X}_{\sigma(p)} + f(p)\sigma_* X_p \\ &= f(p)\nabla_X\sigma(p)\end{aligned}$$

(2) The second relation is a direct consequence of (2)
(3) The third relation comes from proposition (9.2) and the definition of ∇;

$$\begin{aligned}\nabla_X(f\sigma) &= \frac{d}{dt}\Big|_{t=0} (Fl^{\tilde{X}}_{-t} \circ f\sigma \circ Fl^X_t(p)) \\ &= \frac{d}{dt}\Big|_{t=0} \left(Fl^{\tilde{X}}_{-t} \circ f(\gamma_p(t))\, \sigma(\gamma_p(t))\right) \\ \text{by prop (9.2)} &= \frac{d}{dt}\Big|_{t=0} f(\gamma_p(t)) Fl^{\tilde{X}}_{-t}(\sigma(\gamma_p(t)) \\ & (X_p \cdot f)\, \sigma(p) + f(p)\, (\nabla_X\sigma)(p)\end{aligned}$$

(4) The last relation again comes from the fact that Fl^X_t is linear

□

Definition 9.5. A linear connection on a vector bundle is a choice of a linear isomorphism among fibers, equivalently a linear horizontal lift.

We are ready now to work in coordinates. Consider the coordinates functions $(x^1, .., x^n)$ for M, the coordinate basis for T_pM, and the trivialization map Φ associated to the local frame (e_α) with $\alpha = 1, .., k$; writing the vector field $X = a^i\partial_i$ the section as $\sigma = \sigma^\alpha e_\alpha$, we have using (9.4)

$$\begin{aligned}\nabla_X\sigma &= \nabla_{a^i\partial_i}(\sigma^\alpha e_\alpha) \\ \text{by (1)} &= a^i\nabla_i(\sigma^\alpha e_\alpha) \\ \text{by (3)} &= a^i\partial_i(\sigma^\alpha)e_\alpha + a^i\sigma^\alpha\nabla_i e_\alpha\end{aligned}$$

where $\nabla_{\partial_i} := \nabla_i$. Let's focus now on the element $\nabla_i e_\alpha$. Due to the linearity of the construction we have that

$$\nabla_i e_\alpha = B(\partial_i)^\beta{}_\alpha e_\beta$$

for some objects $B(\partial_i)^\beta{}_\alpha$ we now analyze. Fixed the index i this is nothing else that a $k \times k$ matrix associated to an endomorphism of $\mathbb{R}^k$. Fixed the indices α, β and due to the first two of (9.4) we view $B^\beta{}_\alpha$ as maps $\mathfrak{X}(M) \to C^\infty(M)$ linear over $C^\infty(M)$. In conclusion we have that locally $B^\beta{}_\alpha \in \Omega^1(M, End(\mathbb{R}^k))$. We will call it connection one form and just denote its components by $(B_i)^\beta{}_\alpha$ or simply $B_i{}^\beta{}_\alpha$ and call them Christoffel symbols. It is common and useful to write

$$\nabla_i \sigma^\alpha = \partial_i \sigma^\alpha + B_i{}^\alpha{}_\beta \sigma^\beta$$

Observe now that this construction depends on the choice of coordinates and trivialization. The natural question is what happens when we change trivialization and coordinate chart. Suppose on U we have the coordinate chart $(x^1, .., x^n)$ and local frame $e_1, .., e_k$ while on V we have $(\tilde{x}^1, .., \tilde{x}^n)$ and e $\tilde{e}_1, .., \tilde{e}_k$. We know that $\tilde{\sigma}^\beta = \tau^\beta{}_\alpha \sigma^\alpha$ We want to compare on $U \cap V$ the expressions

$$a^i(\partial_i \sigma^\alpha + B_i{}^\alpha{}_\beta \sigma^\beta) e_\alpha$$

and

$$\tilde{a}^i(\tilde{\partial}_i \tilde{\sigma}^\alpha + \tilde{B}_i{}^\alpha{}_\beta \tilde{\sigma}^\beta) \tilde{e}_\alpha$$

where $\tilde{B}$ is the connection one form one would obtain in the trivialization induced by the frames $\tilde{e}_\alpha$ and the coordinate $\tilde{x}$, that is $\tilde{B}_i{}^\alpha{}_\beta \tilde{\sigma}^\beta \tilde{e}_\alpha = \tilde{B}(\tilde{\partial}_i)^\alpha{}_\beta \tilde{\sigma}^\beta \tilde{e}_\alpha d$. Remembering that $\tilde{\partial}_i = \frac{\partial x^j}{\partial \tilde{x}^i} \partial_j$ and observing that the one form connection $\tilde{B}$ can be viewed as a one form taking values in the algebra of $k \times k$ matrices (i.e. $\mathbb{M}_k$) thus recalling that $\tau : M \to \mathbb{M}_k$ one has

$$\tilde{\partial}_i \tilde{\sigma}^\alpha + \tilde{B}_i{}^\alpha{}_\beta \tilde{\sigma}^\beta = \frac{\partial x^j}{\partial \tilde{x}^i} \left(\partial_j (\tau^\alpha_\gamma \sigma^\gamma) + (\tau)^\beta_\gamma (\tilde{B}_j)^\alpha_\beta \sigma^\gamma \right)$$

from which we get, combining with $e_\alpha = \tau^\beta{}_\alpha \tilde{\epsilon}_\beta$, that

$$(\tilde{B}_i)^\alpha_\beta = \frac{\partial x^j}{\partial \tilde{x}^i} (\tau^{-1})^\alpha_\delta (B_j)^\delta{}_\gamma \tau^\gamma{}_\beta + \frac{\partial x^j}{\partial \tilde{x}^i} (\tau^{-1})^\alpha_\delta \partial_j \tau^\delta{}_\beta$$

This ugly formula is often written in a compact form, suppressing the matrix indices and using differential forms notation, as

$$\tilde{B} = \tau^{-1} B \tau + \tau^{-1} d\tau$$

When dealing with TM instead of E we have a natural coordinate basis for the fiber $\{\partial_i\}$ and we denote the connection one form by $\Gamma^j{}_k = (\Gamma_i)^j{}_k dx^i = \Gamma_i{}^j{}_k dx^i$. In this case we have $\tau^i{}_j = \tilde{\partial}_i x^j$ and the connection is called affine connection. We have defined the covariant derivative associated to an affine connection on vector fields only so far. We now generalize to every tensor field $T \in \mathcal{T}^k_l M$ by the following.

(1) $k = 0, l = 1$ we have that $\nabla_X Y$ is defined as before, in components by

$$\nabla_i Y^j = \partial_i Y^j + \Gamma_i{}^j{}_k Y^k$$

(2) $k = 0 = l$ we define it as

$$\nabla_X f = X \cdot f$$

(3) $T = F \otimes G$ then

$$\nabla_X T = \nabla_X F \otimes G + F \otimes \nabla_X G$$

(4) Denoting by Y_i and ω^j with $i = 1, .., k$ and $j = 1, .., l$ two sets of vector fields and covectors (NOT the components of a (co)vector field) we require

$$\begin{aligned}(\nabla_X)F(Y_1, .., Y_k, \omega^1, .., \omega^l) &= X \cdot F(Y_1, .., Y_k, \omega^1, .., \omega^l)\\ &-F(\nabla_X Y_1, Y_2, ..) - ..\\ &-F(Y_1, .., Y_k, \nabla_X \omega^1, ..) - ..\end{aligned}$$

Observe that the extension is unique. We can in fact define the covariant derivative on covectors by

$$(\nabla_X \omega)(Y) = X \cdot \omega(Y) - \omega(\nabla_X Y)$$

and once we know it again using the number (4) we can uniquely extend the construction to every tensor field.In components in the coordinate basis we have

$$\nabla_i \omega_j = \partial_i \omega_j - \Gamma_i{}^k{}_j \omega_k$$

and in general

$$\begin{aligned}\nabla_i T^{j_1..j_l}{}_{m_1..m_k} &= \partial_i T^{j_1..j_l}{}_{m_1..m_k}\\ &+\Gamma_i{}^{j_1}{}_j T^{j..j_l}{}_{m_1..m_k} + ..\\ &-\Gamma_i{}^m{}_{m_1} T^{j_1..j_l}{}_{m..m_k} - ...\end{aligned}$$

It is often useful to work along curves. Consider $\gamma : I \to M$ a smooth curve and define a vector field along γ as a smooth map $\tilde{Y} : I \to TM$ s.t. $\tilde{Y}(t) \in T_{\gamma(t)}M$. A vector field along γ is called extendible if exists a vector field Y at least in an open subset of M containing γ, such that $Y(\gamma(t)) = \tilde{Y}(t)$. This construction naturally extends to all tensor fields. An affine connection induces on the space of vector field along a curve a unique linear operator $D_t : \mathfrak{X}(\gamma) \times \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$ satisfying $D_t(f\tilde{Y}) = \dot{f}\tilde{Y} + fD_t\tilde{Y}$Consider then an extendible vector field along γ and define

$$D_t\tilde{Y}(t) := \nabla_t Y := \nabla_{\dot{\gamma}(t)} Y$$

In components we have

$$\nabla_{\dot{\gamma}(t)} Y = \underbrace{\dot{\gamma}^i \partial_i Y^k}_{\dot{Y}^k}(\gamma(t))\partial_k|_{\gamma(t)} + \dot{\gamma}^i \Gamma_i{}^k{}_j Y^j \partial_k|_{\gamma(t)} \tag{3}$$

In the following we will deal with vector field along curves obtained by restriction thus extendable.

Definition 9.6. A section σ is said to be parallel transported along $\gamma(t)$ with $t \in I$ if $\nabla_{\dot{\gamma}}\sigma = 0$ for every $t \in I$

Observe that by (2) parallel transport and horizontal lift are on the same footing. Suppose we are given an element $\boldsymbol{\rho} \in E_p$ on the fiber over $p = \gamma(0)$. The parallel transport of $\boldsymbol{\rho}$ along γ (with $\gamma(0) = p$) is the unique local section $\sigma \in \Gamma(E)$ s.t.

$$\nabla_{\dot{\gamma}}\sigma = 0$$
$$\sigma(p) = \rho$$

and in this sense parallel transport induce a way of moving elements of the fibers along a curve, and this provides linear isomorphisms between the fibers at points along the curve (by the properties of the covariant derivative that one has to assume from this point o view).

Comment: Often textbook uses a dual approach defining the parallel transport and then the fiber isomorphism using the covariant derivative operator.

Definition 9.7. A smooth curve is called a geodesics if its tangent vector is parallel transported along itself that is $D_t\dot{\gamma} = 0$. In components, by (3) we get

$$\ddot{\gamma}^k + \dot{\gamma}^i \Gamma_i{}^k{}_j \dot{\gamma}^j = 0$$

Theorem 9.8. *Consider an affine connection ∇ defined on M. For every $p \in M$ and $Y_p \in T_pM$ there exist an open interval $I \subseteq \mathbb{R}$ containing 0 and a geodesices $\gamma : I \to M$ satisfying $\gamma(0) = p$ and $\dot{\gamma}(0) = Y_p$. Any two such geodesics agree on their common domain.*

Proof. (*) Sketchy this is a consequence of existence of uniqueness of the solution of an ODE. Consider a coordinate chart (U, φ) with coordinate for $\gamma(t)$ given by $(x^1(t), .., x^n(t))$ and define then the velocity by $v^i = \dot{\gamma}^i$. Then the geodesics equation yields

$$\dot{x}^i = v^i$$

and

$$\dot{v}^k = -v^i v^j \Gamma_i{}^k{}_j(x(t))$$

Thinking at (x, v) as coordinate on $(U \times \mathbb{R}^n)$the previous equation is just the flow induced by the vector field on $U \times \mathbb{R}^n$

$$v^k \frac{\partial}{\partial x^k} - v^i v^j \Gamma_i{}^k{}_j(x(t)) \frac{\partial}{\partial v^k}$$

Then by the existence and uniqueness (locally) of the solution of an ODE the statement follows. □

9.1. **Riemannian geometry.** A Riemannian metric on a manifold M is the assignment to each point p in M of an inner product on the tangent space T_pM that is required to be smooth. Thus we can view a Riemannian metric as a symmetric covariant tensor $g \in \Sigma^2 M$ that is

$$g(X, Y) = g(Y, X)$$

that pointwise induces an inner product $< \bullet, \bullet >_p$ on T_pM. For every vector fields X, Y it is sometimes useful to write

$$< X, Y >_p = g(X, Y)(p) = g_p(X_p, Y_p)$$

Given ω and α one form it is sometimes useful to write their symmetric product as follows

$$\omega\alpha := \frac{1}{2}(\omega \otimes \alpha + \alpha \otimes \omega)$$

With this notation in mind we write the metric in a given coordinate chart as

$$g = g_{ij} dx^i \, dx^j$$

The pair (M, g) is called Riemannian manifold

Warning: A common mistake made by novices is to assume that one can find coordinates near p such that the coordinate vector fields ∂_i are orthonormal. The coordinate basis for T_p is a canonical choice but sometimes not the best one. At each point for example one can find an orthonormal basis. We define then an othonormal frame as a set of sections of TM $(e_1, .., e_p)$ such that

$$g(e_a, e_b) = \delta_{ab} = \begin{pmatrix} 1 & 0 & .. & 0 & 0 \\ 0 & 1 & ... & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & ... & 1 & 0 \\ 0 & 0 & ... & 0 & 1 \end{pmatrix}$$

Dually we a have the orthonormal coframe (pointwise basis for T^*_pM) $(\epsilon^1, .., \epsilon^n)$ where by construction $\epsilon^b(e_a) = \delta^b_a$. Thus we could write in this basis

$$g = \delta_{ab}\epsilon^a\epsilon^b$$

We could always write e_a as linear combination over $C^\infty M$ of ∂_i that is

$$e_a := e^i_a\partial_i$$

as well as

$$\epsilon^a = e^a_i dx^i$$

for some functions (or matrices if you prefer) e^i_a and e^b_j that are such that

$$e^a_i e^i_b = \delta^a_b\,, \qquad e^i_a e^a_j = \delta^i_j$$

being e_a and ϵ^a (as well as dx^i and ∂_i) pointwise dual basis. Observe that by construction we have

$$g = \delta_{ab}\epsilon^a\epsilon^b = \left(\delta_{ab}e^a_i e^b_j\right) dx^i dx^j$$

thus

$$g_{ij} = \delta_{ab}e^a_i e^b_j\,, \quad \text{and viceversa} \quad \delta_{ab} = g_{ij}e^i_a e^j_b\,. \tag{4}$$

Those objects are the main ingredients of the Cartan moving frame theory and other physical models like supergravity and superstring theory, because, in some sense, they encode the information of the metric in a more geometrical way.

Definition 9.9. Take now 2 Riem, manifolds (M, g) and (M', g'), a diffeomorphism $F : M \to M'$ is called an isometry if $F^*g' = g$ (and we will say that (M, g) and (M', g') are isometric). If for every $p \in M$ we can find a neighborhood U of p such that a map $F|_U$ is an isometry then we will call F a local isometry and the two manifolds locally isometric. A Riemannian manifold is named FLAT if it is locally isometric to the Euclidean space.

Theorem 9.10. *On every manifold M there is a Riemannian metric.*

Proof. This metric can be constructed as follows; given a coordinate chart (U, φ), at each point $p \in U$ we can define the inner product

$$< \partial_i|_p, \partial_j|_p >= \delta_{ij}$$

and thus we have a metric on U. Given an Atlas we can define on each chart a Riemannian metric and mix everything using the partition of unity subordinate to the given atlas. □

We now see ho to construct Riemannian metrics in certain natural situations.

Lemma 9.11. *Suppose (M', g') is a Riemannian manifold, M is a smooth manifold and $F : M \to M'$ is a smooth map. Then $g := F^*g'$ is a Riemannian metric on M if and only if F is an immersion.*

(*) Suppose (M', g') is a Riemannian manifold: given a smooth immersion $F : M \to M'$, then we can construct an induced metric on M by $g := F^*g'$. On the other hand, if M is already endowed with a given Riemannian metric g, an immersion or embedding $F : M \to M'$ satisfying $F^*g' = g$ is called an isometric immersion or isometric embedding.
In the case we deal with embedded or immersed submanifolds $M \subset M'$ (that is $F : N \to M'$ with $F(N) = M$ is an immersion or embedding) then we will say that M equipped with the induced metric obtained by the inclusion map $\iota : M \to M'$ is a Riemannian submanifold. Let's see now how to get the induced metric from a m dimensional Riemannian manifold M' to an n dimensional submanifold M. Consider the smooth map $X : \widetilde{U} \to M'$ with $\widetilde{U}$ open subset in $\mathbb{R}^n$ so that $X(\widetilde{U})$ is an open subset of M and such that X is a diffeo into its image we call U. Then its inverse $\varphi : U \to \tilde{U}$ can be viewed as a coordinate map that is (U, φ) is a chart for M with coordinates functions $(x^1, .., x^n)$. This is called local parametrization. Then denoting by g the induced metric by the inclusion map by g' we have

$$X^*g = X^*\iota^*g' = (\iota \circ X)^*g' = X^*g'$$

In the case $M' = \mathbb{R}^m$ equipped with the standard Euclidean metric, the induced metric on U would be given by

$$g = X^*(\sum_{i=1}^{m}(dy^i)^2 = (d(y^i \circ X))^2 = (dX^i)^2 = \sum_{i=1}^{m}(\sum_{j,k=1}^{n} \frac{\partial X^i}{\partial x^j}dx^j \frac{\partial X^i}{\partial x^k}dx^k)$$

Musical isomorphism:

Definition 9.12.

$$\flat : TM \to T^*M$$

by $\flat(X) := X^\flat$ is the covector such that $X^\flat(Y) = g(X, Y)$ for every $Y \in TM$

Working in component we have $X_i := X^\flat(\partial_i) = g_{ij}X^j$. Being the metric invertible one can also define g^{-1} that in components looks like $g^{-1} = g^{ij}\partial_i \otimes \partial_j$ with $g^{ij} = g^{ji}$ where $g_{ij}g^{jk} = \delta_i^k$. This induces an inner product on the cotangent space T_p^*M, namely:

$$< \omega_p, \alpha_p >_p:= g^{-1}(p)(\omega_p, \alpha_p) = g^{ij}(p)\omega_i(p)\omega_j(p)$$

The inverse map of the flat map is given by the sharp map

$$\sharp : T^*M \to TM$$

by $\omega^\sharp$ such that $\alpha(\omega^\sharp) = g^{-1}(\omega, \alpha)$ that is in components

$$dx^i(\omega^\sharp) := \omega^i = g^{ij}\omega_j$$

This operators can be applied to any type of tensor to raise or lower indices.

We know go back to the integvation problem. We consider an orientable n dimensional Riemannian manifold (M, g) and consider positively oriented orthonromal frame and coframe $\{e_a\}$ and $\{\epsilon^a\}$. It is natural to require that the area spanned by the oorthonormal basis is one, we thus want that exist a Riemannian volume form (alias a top form) satisfying, locally

$$dV_g(e_1, .., e_n) = 1$$

This object is unique and given locally simply by

$$dV_g = \epsilon^1 \wedge ... \wedge \epsilon^n$$

It is useful to write everything in terms of coordinate function $(x^1, .., x^n)$ associated to an oriented coordinate chart, and the induced coordinate basis for sections of any tensor bundle. Namely being

$$\epsilon^a = e^a_i dx^i$$

we obtain

$$dV_g = \det e^a_i dx^1 \wedge ... \wedge dx^n$$

by the Binet theorem and (4) we have

$$\det g_{ij} = \det e^a_i \det \delta_{ab} \det e^b_j$$

thus

$$dV_g = \sqrt{\det g_{ij}} dx^1 \wedge ... \wedge dx^n$$

Consider then f a compactly supported functions then $f dV_g$ is a compactly supported top form then

$$\int_M f dV_g$$

is well defined.

When our manifold is not oriented we run into the annoying problem of the change of coordinate formula when the determinant of the Jacobian has not the absolute value as it should. For this reason one define the densities (namely section of the density bundle we skip more details) that are object of the form

$$\mu = u|dx^1 \wedge ... \wedge dx^n|$$

for some smooth function u and where for a top form ω we define

$$|\omega|(X, Y, ..., Z) := |\omega(X, Y, ..., Z)|.$$

Given a smooth function f we can define its gradient by

$$grad(f) := (df)^\sharp$$

Moreover given a vector field X on an oriented n-dimensional Riemannian manifold, we can construct $dV_g(X, \bullet, ..., \bullet) \in \Omega^{n-1}(M)$; taking its exterior derivative we obtain another top form we use to define $divX$ as follows

$$d(dV_g(X)) := (divX)dV_g$$

We can finally defined an important object called the Laplace-Beltrami operator $\Delta : C^\infty(M) \to C^\infty(M)$ defined by

$$\Delta f := div(grad(f))$$

In certain cases it would be useful to define it with a minus sign in order to have non negative eigenvalue for the Laplacian.

9.2. **Riemannian distance.** We define a regular curve as a curve $\gamma : I \to M$ so that $\dot{\gamma}(t) \neq 0$ for $t \in I$, that is γ is an immersion. A curve segment is a curve defined on a compact interval. It is useful to deal with picewise regular curve segment that is a curve segment $\gamma : [a, b] \to M$ so that I can find a partition of $[a, b]$ that is $a_0 < a_1 < .. < a_k$ with $a_k = b$ and $a_0 = a$ so that $\gamma|_{[a_i,a_{i+1}}$ is regular for every $i = 0, .., k-1$. Note that in this definition the speed of the curve approaching a_i from the left and right are not required to be equal. We call those curve admissible curve.

Definition 9.13. Consider an admissible curve $\gamma : [a, b] \to M$ we define its length to be

$$L_g(\gamma) = \int_a^b g(\dot{\gamma}, \dot{\gamma})dt$$

It is easy to check that this definition is parameter independent and invariant under isometry. Moreover one can prove that if M is a connected smooth manifold then any two points can be joined by an admissible curve. Thus for a connected Riemannian manifold we can construct a notion of distance as follows

$$d_g(p, q) = inf L_g(\gamma)$$

where gamma is an arbitrary admissible curve connecting p and q.

Observation Any Riemannian manifold equipped with this notion of distance is a metric space.

Proposition 9.14. *Let (M, g) be a connected Riemannian manifold. With the distance function defined above M is a metric space whose induced topology is the same as the given manifold topology.*

It is convenient to become confident with clever reparametrization of a regular curve. If $\gamma : I \to M$ is a curve and $\varphi : \tilde{I} \to I$ is a diffeomorphism of intervals in $\mathbb{R}$ then define the reparametrization of γ by $\tilde{\gamma} : \gamma \circ \varphi$. Consider now a curve in a Riemannian manifold (M, g) and an arbitrary t_0 in I and construct

$$s(t) = \int_{t_0}^t \sqrt{g(\dot{\gamma}(\tau), \dot{\gamma}(\tau))}d\tau$$

by construction $\dot{s} = \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))} > 0$ thus we can interpret s as a diffeo from $I \to \tilde{I}$. Define now a reparametrization of gamma as follows: $\tilde{\gamma} = \gamma \circ s^{-1}$. It is easy to see by applying the chain rule that $\tilde{\gamma}$ has unit speed that is $g(\dot{\tilde{\gamma}}, \dot{\tilde{\gamma}}) = 1$. We summarize in the following

Proposition 9.15. *Every regular curve in a Riemannian manifold has a unit-speed reparametrization.*

9.3. **Metric connection.** Give a Riemannian manifold we want to describe the compatibility between the metric and the connection. To this aim we say that

Definition 9.16. An affine connection on Riemannian manifold is compatible with the metric if given a curve γ and two vector fields X and Y parallel transported along γ we have that $g_{\gamma(t)}(X_{\gamma(t)}, Y_{\gamma(t)})$ is constant. We will call it metric connection

Let's now characterize the metric connection

Proposition 9.17. *The following are equivalent:*

(1) The connection is compatible with the metric

(2) X,Y vector fields along a curve then $\frac{d}{dt}g(X,Y) = g(D_tX,Y) + g(X,D_tY)$
(3) X,Y,Z *vector fields than* $X \cdot g(Y,Z) = g(\nabla_x Y, Z) + g(X, \nabla_X Y)$
(4) $\nabla_X g = 0$ *for every* X

Proof. 1) $\to$ 2) : If the connection is compatible with the metric we can take an orthonormal basis on a point $p \in \gamma$ and then construct their parallel transported vector fields $(e_1, .., e_n)$ (NOT that general ODE theory in fact assure us that the solution of the parallel trnasport equation with some given initial condition exsits and is unique!!). Being the connection compatible with metric the inner product is constant along the curve thus $(e_1, .., e_n)$ is an orthonormal frame along the curve. Using it we have locally

$$X = X^a e_a\,, \quad Y = Y^b e_b$$

from which

$$D_t X = (\partial_t X^a) e_a$$

since $D_t e_a = 0$ by assumption. Thus

$$g(D_tX,Y) + g(X,D_tY) = \dot{X}^a Y^b \delta_{ab} + \dot{Y}^a X^b \delta_{ab} = \frac{d}{dt}g(X,Y)$$

the other way around follows easily by the assumption. Let's see now how 3) $\to$ 2).Consider

$$\frac{d}{dt}g(X,Y) = (\dot{X}^i Y^j + X^i \dot{Y}^j) g_{ij} + X^i Y^j \frac{d}{dt} g(\partial_i, \partial_j)$$

by using 3) we have

$$\frac{d}{dt} g(\partial_i, \partial_j) = g(D_t \partial_i, \partial_j) + g(\partial_i, D_t \partial_j)$$

and then the result follows. The other way around can be proved along the same line. Let's now see 3) $\leftrightarrow$ 4)

$$(\nabla_X g)(Y,Z) = X \cdot g(Y,Z) - g(\nabla_x Y, Z) - g(\nabla_X Z, Y)$$

and this is zero for every X,Y,Z iff 3) is satisfied for all X,Y,Z. □

Corollary 9.18. *Given a geodesics* γ *on a Riemannian manifold* (M,g) *then* $< \dot{\gamma}, \dot{\gamma} >_g$ *is constant along the curve*

there are too many connection metric compatible; to pin down a unique one we require the following condition:

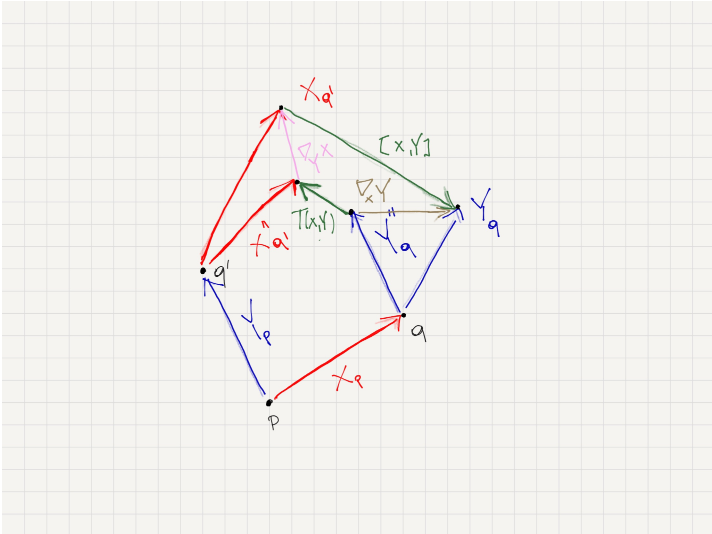
$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] = 0$$

It is easy to show that T is a $(2,1)$ tensor called $TORSION$. An affine connection satisfying this relation is called torsion free or symmetric since torsion freeness implies in components that

$$\Gamma_i{}^k{}_j = \Gamma_j{}^k{}_i =: \Gamma^k_{ij}$$

Graphically, using a "small time approximation", and denoting by $X^{//}$ the vector field obtained by the parallel transport of a tangent vector at a general point p, along some curve γ,we can visualize the torsion as described in diagram below. Note that if X is parallel trnasporte along the integral curve for Y then $\nabla_Y X = 0$, thus in a linear approximation we can identify $X - X^{//}$ with $\nabla_Y X$

Theorem 9.19. *On a Riemannian manifold there is a unique metric compatible symmetric affine connection called the Levi Civita connection*



Proof. Metric compatibility and symmetry gives

$$X \cdot g(Y,Z) \quad = \quad g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

$$\nabla_X Z = \nabla_Z X + [X,Z] \text{ by torsionfreeness} \quad = \quad g(\nabla_X Y, Z) + g(Y, \nabla_Z X) + g(Y, [X,Z])$$

by permuting the previous result on the symbols X, Y, Z in a clever way we have

$$X \cdot g(Y,Z) + Y \cdot g(Z,X) - Z \cdot g(X,Y) = 2g(\nabla_X Y, Z) + g(Y,[X,Z]) + g(Z,[Y,X]) - g(X,[Z,Y])$$

that we can solve for $g(\nabla_X Y, Z)$:

$$\begin{aligned} g(\nabla_X Y, Z) \quad &= \quad \tfrac{1}{2}\,(X \cdot g(Y,Z) + Y \cdot g(Z,X) - Z \cdot g(X,Y) \\ &\quad -g(Y,[X,Z]) - g(Z,[Y,X]) + g(X,[Z,Y])) \end{aligned}$$

Note that two Levi Civita connections ∇ and $\tilde{\nabla}$ are such that

$$g(\nabla_X Y - \tilde{\nabla}_X Y, Z) = 0$$

being the metric non singular we must have then $\nabla_X Y = \tilde{\nabla}_X Y$. In a given coordinate chart we thus have

$$g(\nabla_i \partial_j, \partial_k) = \frac{1}{2}(\partial_i g(\partial_j, \partial_k) + \partial_j g(\partial_k, \partial_i) - \partial_k g(\partial_i, \partial_j))$$

that is

$$\Gamma^k_{ij} = \frac{1}{2} g^{km}(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij})$$

This formula prove the existence of the Levi Civita connection in a coordinate chart and thus everywhere. □

What happen in a different frame?? Consider the orthonormal frame $\{e_a\}$ and define the smooth functions f^c_{ab} by

$$[e_a, e_b] = f^c_{ab} e_c$$

Then one gets

$$\Gamma^c_{ab} = \frac{1}{2}(f^c_{ab} - f^{b'}_{ac'} \delta^{cc'} \delta_{b'b} - f^{a'}_{bc'} \delta^{cc'} \delta_{a'a})$$

Consider now an embedded surface in $\mathbb{R}^3$ etc...

Suppose M and M' are smooth manifolds and ∇' an affine connection on M' and $F : M \to M'$ is a diffeomorphism then $F^*\nabla'$ defined by

$$(F^*\nabla')_X Y = (F^{-1})_*(\nabla_{F_*X} F_*Y)$$

is an affine connection on M called the pullback connection.

Proposition 9.20 (Naturality of the LC connection)**.** *Suppose (M,g) and (M',g') are Riemannian with Levi Civita connections ∇ and ∇'. If $F : M \to M'$ is an isometry then $F^*\nabla' = \nabla$*

Ledt's now go back to geodesics for the LC connection.

Definition 9.21. Let (M,g) be a Riemannian manifold. An admissible curve γ in M is said to be a minimizing curve if $L_g(\gamma) \leq L_g(\tilde{\gamma})$ for every admissible curve $\tilde{\gamma}$ with the same endpoints.

Now we state the following result whose proof involve calculus of variations tools we are not going to discuss in this notes

Proposition 9.22. *In a Riemannian manifold, every minimizing curve is a geodesic when it is given a unit-speed parametrization.*

From this proposition we have the following result:

Corollary 9.23. *A unit-speed admissible curve is a critical point for $L_g(\gamma)$ if and only if it is a geodesic.*

It is easy to see that the literal converse is not true, because not every geodesic segment is minimizing. For example, every geodesic segment on S^2 that goes more than halfway around the sphere is not minimizing, because the other portion of the same great circle is a shorter curve segment between the same two points. What can be proved by using Riemann normal coordinate is that geodesics γ are locally minimizing that is for every $t_0 \in I$ we can find a neighborhood I_0 such that γ restricted to I_0 is minimizing

9.4. **Geodesics in Riemannian geometry (***).** We have already discussed the existence and uniqueness of a geodesics given the initial speed and point. Now we will focus on the case of geodesics obtained out of the Levi Civita connection only. Let now denote by Υ_p^X the geodesics with initial point p and speed X_p, be carefull this is not the integral curve for X. From the construction it is easy to see the the following relation holds:

$$\Upsilon_p^{aX}(t) = \Upsilon_p^X(at)$$

whenever either side is defined. Consider now $\mathcal{E} = \{X \in \mathfrak{X}(M)$ s.t. Υ_p^X is defined in an interval containing [0,1] for all $p\}$; then construct the following map

$$exp(X) = \Upsilon_\bullet^X(1)$$

or simply $exp_p(X)$ when we want to specify also the initial point

Proposition 9.24. *For each X the geodesics Υ_p^X is given by*

$$\Upsilon_p^X(t) = exp_p(tX)$$

Proof. This is a direct consequence of the rescaling property discussed before □

When we restrict to a point is sometimes useful to denote by $\mathcal{E}_p$ vectors at p so that the exponential map is well defined.

Proposition 9.25. *(*) Let $F : M \to N$ be an isometry of Riemannian manifolds and $p \in M$. Then the following diagram is commutative*

$$\begin{array}{ccc} \mathcal{E}_p & \xrightarrow{dF|_p} & \mathcal{E}_{F(p)} \\ \downarrow{\scriptstyle exp_p} & & \downarrow{\scriptstyle exp_{F(p)}} \\ M & \xrightarrow{F} & N \end{array}$$

Consider now $T_{\mathbf{0}}(T_pM)$; it can be canonically identified with T_pM. If we consider the differential of the exponential map at the origin that is $d(exp_p)|_{\mathbf{0}}$ it can be viewed as a map from T_pM to itself. Remember (or oserve) that giving a map $F : M \to N$ one has that $F_*(\dot{\gamma}) = (F \dot{\circ} \gamma)$. Consider now a curve on T_pM with some initial tangent vector X_p and starting at $\mathbf{0}$ for example $c(t) = tX_p$ and use it to compute the differential of the exponential map:

$$(exp_p)_*|_{\mathbf{0}}(X_p) = d(exp_p)|_{\mathbf{0}}(X_p) = \frac{d}{dt}|_0 exp_p(c(t)) = \frac{d}{dt}|_0 exp_p(tX_p) = \frac{d}{dt}|_0 \Upsilon_p^{X_p}(t) = X_p$$

We reassume this result in the following proposition

Proposition 9.26. *The differential at the origin of the exponential map is the identity map on T_pM*

This result is crucial since by the inverse function theorem we can say that there is a local diffeomorphism from T_pM to M around the origin. Better to say , there are neighborhoods V of $\mathbf{0}$ in T_pM and U of p in M such that $exp_p : V \to U$ is a diffeomorphism. A neighborhood U of p that is diffeomorphic trough the exponential map to some neighborhood V of $\mathbf{0}$ in T_pM is called a NORMAL NEIGHBORHOOD of p. Given an orthonormal basis $e_i|_p$ for T_pM define the map $\Phi : \mathbb{R}^n \to T_pM$ associated to the trivialization induced pointwise by the orthonormal frame by

$$\Phi(y^1, .., y^n) = y^i e_i|_p$$

Being the exponential map locally invertible we can construct for a normal neighborhood of p the clever coordinate chart (U, φ), called normal coordinate centered at p by constructing φ as follows

$$\begin{array}{ccc} V \subset T_pM & \xrightarrow{\Phi^{-1}} & \mathbb{R}^n \\ {\scriptstyle (exp_p|_V)^{-1}}\uparrow & \nearrow{\scriptstyle \varphi} & \\ U & & \end{array}$$

Observe that given this coordinate we have that the coordinate basis for T_pM is orthonormal so in some sense we combine the beauty of the coordinate and orthonormal frame. Nice things happens in this coordinates:

Proposition 9.27. *Given a Riemannian manifold (M, g) and a normal coordinate chart centered at p we have*

(1) In this coordinate chart $g_{ij}(p) = \delta_{ij}$
(2) For every $X = a^i\partial_i$ the geodesics Υ_p^X is represented in this coordinates by the line $(ta^1, .., ta^n)$
(3) the Christoffel symbols at p in this coordinates vanish

Proof. □

From this proposition it is evident that locally Riemannian geodesic substitute the notion of straight lines. One can easily prove that for S^2 grate circles are geodesics.

9.5. **The Riemannian curvature.** We look for a mathematical object telling us if a Riemannian manifold is flat (locally isometric to the Euclidean space or not).We note that on an Euclidean space given a vector field Z one has, for example, that

$$\nabla_1 \nabla_2 Z - \nabla_2 \nabla_1 Z = \partial_1 \partial_2 Z - \partial_2 \partial_1 Z = 0$$

or more in general that

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{[X,Y]} \cdot Z$$

and this motivate the following definition:

Definition 9.28.

Given a Riemannian manifold we define the (3,1) tensor field R, called the Riemann tensor, by the following

$$R(X, Y, Z) = R(X, Y) \cdot Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

with ∇ being the Levi Civita connection and where we interpret $R(X, Y)$ as a (1,1) tensor field thus pointwise an element of $End(T_pM)$ and for this reason one can call it the curvature endomorphism. In components one has

$$R(\partial_i, \partial_j, \partial_k) = R^l{}_{kij} \partial_l$$

or if you prefer

$$R^l{}_{kij} = dx^l (R(\partial_i, \partial_j, \partial_k))$$

In components one gets explicitly

$$R^l{}_{kij} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^m_{jk} \Gamma^l_{im} - \Gamma^m_{ik} \Gamma^l_{jm}$$

We say that a vector field is parallel with respect to a connection ∇ if it is parallel transported along any curve. We have the following useful Lemma

Lemma 9.29. *Suppose M is a smooth manifold, and ∇ is any flat connection on M. Given $p \in M$ and any vector $\mathbf{v} \in T_pM$, there exists a parallel vector field X (parallel transported along any curve γ) defined on a neighborhood of p such that $X(p) = X_p = \mathbf{v}$.*

Proof. We take a coordinate chart for M centered at p and we assume without loosing generality that its image is a cube on $\mathbb{R}^n$. We will denote by γ_i the integral curve for ∂_i. We parallel transort $\mathbf{v}$ by γ_1 then the results parallel transported by γ_2 and so on. The resulting vector field we call X is such that $\nabla_1 X = 0$ on the x_1 axis (the point where $x_2 = ... = x_n = 0$) $\nabla_2 X = 0$ on the x_1-x_2 plane (the point where $x_3 = ... = x_n = 0$) and so on. In general we have that $\nabla_k X = 0$ on M_k that is the set of points where $x_{k+1} = ... = x_n = 0$. We prove that by induction

$$\nabla_1 X = ... = \nabla_k X = 0 \quad \text{on } M_k$$

It is obviously true for$k = 1$ and we assume that it is true forsome k. On M_{k+1} we have that $\nabla_{k+1} X = 0$ on M_{k+1} and $\nabla_i X = 0$ for $0 \leq i \leq k$ on M_k. Since partial derivative commutes we have that the flatness criterion implies that

$$\nabla_{k+1} \nabla_i X = \nabla_i \nabla_{k+1} X$$

the left hand sides vanishes M_{k+1} and according to the previous formula then $\nabla_i X$ is parallel along the curves γ_{k+1} starting on M_k. But $\nabla_i X = 0$ on M_k and the parallel transport of the zero vector is the zero vector field. Thus $\nabla_i X = 0$ on M_{k+1} for all $i = 1, ..., k + 1$ □

Proposition 9.30. *A Riemannian manifold is flat if and only if its curvature tensor vanishes identically.*

Proof. On the Euclidean space we have that the curvature vanishes thus by the naturality of the Levic Civita connection one has one direction of the statement. Suppose now that the Riemann tensor vanishes identically and thus the Levi Civita connection is such that

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = 0$$

Consider an orthonormal basis $(e_1|_p, ..., e_n|_p)$ for T_pM. Thanks to the previous Lemma we may extend each $e_i|_p$ locally to a parallel vector field $(e_1, ..., e_n)$ with $e_i(p) = e_i|_p$ and since the parallel transport preserve the inner product then $(e_1, ..., e_n)$ is an orthonormal frame. Being the Levi Civita connection torsion free by the definition of parallel vector field we have:

$$\nabla_{e_i} e_j - \nabla_{e_j} e_i - [e_i, e_j] = 0 = [e_i, e_j]$$

Being this frame commutative we can find a coordinate system (U, φ) with coordinates functions (x^i) such that $e_i = \partial_i$ (we should prove it but we skip it). Remember that when we write $\partial_i|_q$ we mean $(\varphi^{-1})_*(\partial_i|_{\varphi(q)})$ where q is any point in U even if we often avoid this notation its hould be clear fromt he context. Denoting by g the metric on M and by $\bar{g}$ the Euclidean metric, by construction we have

$$g(\partial_i, \partial_j) = (g((\varphi^{-1}))_*(\partial_i), (\varphi^{-1})_*(\partial_j) = (\varphi^{-1})^* g(\partial_i, \partial_j)) = \delta_{ij} = \bar{g}(\partial_i, \partial_j)$$

thus $(\varphi^{-1})^* g = \bar{g}$ and φ is the local isometry we were looking for. Note that the ∂_i on the first and last braket of the previous relation are different objects, the first one is a vector field on M the other one on the Euclidean space. ☐

The Riemann tensor satisfies many identities one can easily recover out of its definition. We list them in the following by using the components notation and using the (4,0) tensor obtained by the Riemann one raising an index $R_{ijkl} = g_{im} R^m{}_{jkl}$:

- $$R_{ijkl} = -R_{jikl} \qquad R_{ijkl} = -R_{ijlk}$$
- $$R_{ijkl} = R_{klij}$$
- First Bianchi idenity
$$R_{ijkl} + R_{kijl} + R_{jkil} = 0$$
- Second Bianchi identity
$$\nabla_m R_{ijkl} + \nabla_i R_{jmkl} + \nabla_j R_{mikl} = 0$$

Other interesting tensors are the Ricci tensor Ric that is a $(2, 0)$ symmetric tensor defined in components as

$$R = R_{ij} dx^i dx^j$$

with $R_{ij} = R^k{}_{ikj}$ and the scalar curvature S that is its trace $S = g^{ij} R_{ij}$. A Riemannian manifold is called an Einstein manifold if

$$R_{ij} = \lambda g_{ij}$$

for some $\lambda \in \mathbb{R}$

Definition 9.31. A vector field X is named Killing vector field if $\mathcal{L}_X g = 0$

Proposition 9.32. *If $X = a^k \partial_k$ is a Killing vector field then $\nabla_i a_j + \nabla_j a_i = 0$*

10. Principal bundle and gauge theories (**)

A principal bundle is defined as a fiber bundle $\pi : E \to M$ with standard fiber a Lie group G which is endowed with an equivalence class of principal bundle atlases that we will describe below. The group G is referred to as the structure group of the principal bundle and principal bundles with structure group G are also called principal G-bundle. In the theoretical physics literature G is name instead the gauge group. More in details we have the following

Definition 10.1. A G principal bundle is a fiber bundle $\pi : E \to M$ where E is equipped by a right G action

$$\begin{array}{rcl} R_\bullet(\,) : E \times G & \to & E \\ (P, g) & \mapsto & P \cdot g := R_g(P) \end{array}$$

that along the fiber is free ($P \cdot g = P \Rightarrow g = e$) and transitive ($\forall P, P' \in M \, \exists g \in G \; s.t. \; P' = P \cdot g$) (this definition essentially implies that the each fiber is homeomorphic as topological space to the structure group). Moreover we require that the trivialization map is G equivariant, namely, being $\{U_\alpha\}$ an open cover of M we have that the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\psi_\alpha} & U_\alpha \times G \\ \downarrow \pi & \swarrow pr_1 & \\ U_\alpha & & \end{array}$$

and the trivialization map (that is a diffeomorphism by definition) $\psi_\alpha(P) = (p; = \pi(P), g_\alpha(P))$ is such that $g_\alpha(P \cdot h) = g_\alpha(P) \cdot h$

Let's now discuss the compatibility of trivialization map on the overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Given two trivializations $\psi_\alpha(P) = (p; = \pi(P), g_\alpha(P))$ and $\psi_\beta(P) = (p; = \pi(P), g_\beta(P))$,restricting our analysis on $P \in U_{\alpha\beta}$ we can easily find the transition function $g_{\alpha\beta}$ sucht that $g_\alpha(P) = g_{\alpha\beta}(P) g_\beta(P)$; by the grouppropety in fact $g_\alpha(P) g_\beta^{-1}(P) = g_{\alpha\beta}(P)$. Observe now that the transition function only depends on the fiber not on the point on the fiber in fact the equivariance implies that

$$g_{\alpha\beta}(P \cdot h) = g_\alpha(P) h (g_\beta(P) h)^{-1} = g_\alpha(P) h h^{-1} g_\beta^{-1}(P) = g_{\alpha\beta}(P)$$

thus we view $g_{\alpha\beta}$ as a map from $U_{\alpha\beta}$ and the group G. One cna alsoeasily check

$$g_{\alpha\beta} g_{\beta\alpha} = e = g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha}$$

Definition 10.2. A connection on a G principal bundle $E \to M$ is an equivariant choice of the horizontal bundle, that is at each point $P \in E$ we have a $T_P E = V_P E \oplus H_P E$ with $(R_h)_* H_P = H_{P \cdot h}$.

The action of G on E defines the foundamental vector fields $\xi^\bullet : \mathfrak{g} \to \mathfrak{X}(E)$ by

$$\xi_p^X := \frac{d}{dt}|_{t=0} (P \cdot e^{tX})$$

It is easy to check that $\pi_*(\xi_p^X) = 0$ thus it defines a vertical vector (fields) and since G acts freely we have the isomorphism $\mathfrak{g} \to V_P E$

Lemma 10.3.

$$(R_g)_* \xi^X = \xi^{g^{-1} X g}$$

The horizontal subspace being a linear subspace, can be obtained by $k = dimG$ linear equations $T_PE \to \mathbb{R}$ that is H_PE is the kernel of a one-form ω at P with values in a k dimensional vector space. There is a natural such vector space, namely the Lie algebra $\mathfrak{g}$ of G, and since the one form ω annihilates horizontal vectors it is defined by what it does to the vertical vectors. This yields to the following

Definition 10.4. The connection one form associated to the horizontal bundle HE is the $\mathfrak{g}$ valued one form on E defined by

$$\omega(\mathbf{V}) = \begin{cases} X & \text{if} \quad \mathbf{V} = \xi^X \\ 0 & \text{if} \quad \mathbf{V} \in HE \end{cases}$$

Proposition 10.5. $(R_g)^*\omega = ad_{g^{-1}} \circ \omega$

Proof.

$$(R_g)^*\omega(\xi^X) = \omega((R_g)_*\xi^X) = \omega(\xi^{ad_{g^{-1}}X}) = ad_{g^{-1}}X$$

□

Given a local section $\sigma_\alpha : U_\alpha \to E$ we can pullback ω and define

$$A_\alpha := (\sigma_\alpha)^*\omega|_{U_\alpha} \in \Omega^1(U_\alpha, \mathfrak{g})$$

Proposition 10.6. *Let for simplicity denote $g := g_{\alpha\beta} : U_{\alpha\beta} \to G$; suppose we get in two overlapping trivialization A_α and A_β. then we have, for matrix Lie goup:*

$$A_\alpha = gA_\beta g^{-1} - (dg)g^{-1}$$